

Space-time derivative estimates of the Kock-Tataru solutions to the nematic liquid crystal system in Besov spaces *

Qiao Liu [†]

Department of Mathematics, Hunan Normal University, Changsha, Hunan 410081, P. R. China

June 13, 2014

Abstract

In recent paper [7] (Y. Du and K. Wang, Space-time regularity of the Kock & Tataru solutions to the liquid crystal equations, *SIAM J. Math. Anal.*, **45**(6), 3838–3853.), the authors proved that the global-in-time Koch-Tataru type solution (u, d) to the n -dimensional incompressible nematic liquid crystal flow with small initial data (u_0, d_0) in $BMO^{-1} \times BMO$ has arbitrary space-time derivative estimates in the so called Koch-Tataru space norms. The purpose of this paper is to show that the Koch-Tataru type solution satisfies the decay estimates for any space-time derivative involving some borderline Besov space norms. More precisely, for the global-in-time Koch-Tataru type solution (u, d) to the nematic liquid crystal flow with initial data $(u_0, d_0) \in BMO^{-1} \times BMO$ and $\|u_0\|_{BMO^{-1}} + \|d_0\|_{BMO} \leq \varepsilon$ for some small enough $\varepsilon > 0$, and for any positive integers k and m , one has

$$\|t^{\frac{k}{2}+m}(\partial_t^k \nabla^m u, \partial_t^k \nabla^m \nabla d)\|_{\tilde{L}^\infty(\mathbb{R}_+, \dot{B}_{\infty, \infty}^{-1}) \cap \tilde{L}^1(\mathbb{R}_+, \dot{B}_{\infty, \infty}^1)} \leq \varepsilon.$$

Furthermore, we shall give that the solution admits an unique trajectory which is Hölder continuous with respect to space variables.

Keywords: nematic liquid crystal flow; Navier-Stokes equations; regularity; Littlewood-Paley decomposition; Besov space; trajectory

2010 AMS Subject Classification: 76A15, 35B65, 35Q35

1 Introduction

Liquid crystal refer to a state of matter that has properties between those of a solid crystal and those of an isotropic liquid. Examples of liquid crystals can be found both in nature (e.g., solutions of soap and detergents) and in technological applications (e.g., electronic displays). There are many different types of liquid crystal phases, and the most common phase is the nematic. In a nematic phase, the molecules do not exhibit any positional order, but they have long-range orientational order. The hydrodynamic theory of the nematic liquid crystals, due to Ericksen and Leslie, was developed during the period of 1958 through 1968 (see [9, 21]). Since then, many remarkable developments have been made from both theoretical and applied aspects.

For any $n \geq 2$, the hydrodynamic flow of nematic liquid crystals in $\mathbb{R}_+^{n+1} := \mathbb{R}^n \times \mathbb{R}_+$ is given by

$$\partial_t u - \nu \Delta u + (u \cdot \nabla) u + \nabla P = -\lambda \nabla \cdot (\nabla d \odot \nabla d) \quad \text{in } \mathbb{R}_+^{n+1}, \quad (1.1)$$

$$\partial_t d + (u \cdot \nabla) d = \gamma(\Delta d + |\nabla d|^2 d) \quad \text{in } \mathbb{R}_+^{n+1}, \quad (1.2)$$

*The author is supported by the Hunan Provincial Natural Science Foundation of China (13JJ4043) and National Natural Science Foundation of China (11326155, 11171357).

[†]E-mail address: liuqiao2005@163.com.

$$\nabla \cdot u = 0, \quad |d| = 1 \quad \text{in } \mathbb{R}_+^{n+1}, \quad (1.3)$$

$$(u, d)|_{t=0} = (u_0, d_0), \quad |d_0(x)| = 1 \quad \text{in } \mathbb{R}^n, \quad (1.4)$$

where $u(x, t) : \mathbb{R}_+^{n+1} \rightarrow \mathbb{R}^n$ is the unknown velocity field of the flow, $P(x, t) : \mathbb{R}_+^{n+1} \rightarrow \mathbb{R}$ is the scalar pressure and $d : \mathbb{R}_+^{n+1} \rightarrow \mathbb{S}^2$, the unit sphere in \mathbb{R}^3 , is the unknown (averaged) macroscopic/continuum molecule orientation of the nematic liquid crystal flow, $\nabla \cdot u = 0$ represents the incompressible condition, (u_0, d_0) is a given initial data with $\nabla \cdot u_0 = 0$ in distribution sense, and ν , λ and γ are positive numbers associated to the properties of the material: ν is the kinematic viscosity, λ is the competition between kinetic energy and potential energy, and γ is the microscopic elastic relaxation time for the molecular orientation field. The notation $\nabla d \odot \nabla d = (\partial_i d \cdot \partial_j d)_{1 \leq i, j \leq n}$ is the stress tensor induced by the director field d . Since the concrete values of ν , λ and γ do not play a special role in our discussion, for simplicity, we assume that they all equal to one throughout this paper.

System (1.1)–(1.4) is a simplified version of the Ericksen-Leslie model [9, 21], but it still retains most important mathematical structures as well as most of the essential difficulties of the original Ericksen-Leslie model. A brief account of the Ericksen-Leslie theory on nematic liquid crystal flows and the derivations of several approximate systems can be found in the appendix of [27]. For more details on the dynamic continuum theory of liquid crystals, we refer the readers to the book of Stewart [36]. Mathematically, system (1.1)–(1.4) is a strongly coupled system between the incompressible Navier-Stokes (NS) equations (the case $d \equiv 1$, see e.g., [1–6, 10–12, 17–20, 22, 23, 33, 34, 41]) and the transported heat flows of harmonic map (the case $u \equiv 0$, see e.g., [35, 38]), and thus, its mathematical analysis is full of challenges.

To make a clearer introduction to the results of the present paper, we shall recall some well-posedness and regularity results of the NS equations. In the seminal paper [23], Leray proved the global existence of finite energy weak solutions to (1.1)–(1.3), but its regularity and uniqueness still remain open. The theory of the so called *mild* solutions to the NS equations is pioneered by Fujita and Kato [10, 19], and these works inspired extensive study in the following years on the well-posedness of the NS equations in various critical spaces, see Kato [18], Cannone [3], Koch and Tataru [20], Lemarié-Rieusset [22] and so on. Particularly, Koch and Tataru established the well-posedness for the NS equations with initial data in $BMO^{-1}(\mathbb{R}^n)$. Hereafter, we call the *mild* solution presented by Koch and Tataru [20] as Koch-Tataru solution. For the spatial regularity on the *mild* solution to the NS equations has been studied by many authors, such as Giga and Sawada [12], Sawada [34] and Miura and Sawada [33]. In paper [11], Germain, Pavlović and Staffilani had proved the Koch-Tataru's solution u satisfies the following spatial regularity property:

$$t^{\frac{m}{2}} \nabla^m u \in Z_{T^*} \text{ for all } m \in \mathbb{N},$$

where Z_{T^*} is the Koch-Tataru's solution existence space for the NS equations. In [41], the authors generalized Germain-Pavlović-Staffilani's results, and obtained that for sufficient small enough $u_0 \in BMO^{-1}$, the global-in-time Koch-Tataru's solution satisfies

$$\|t^{\frac{m}{2}} \nabla^m u\|_{\tilde{L}^\infty(\mathbb{R}_+; \dot{B}_{\infty, \infty}^{-1}) \cap \tilde{L}^1(\mathbb{R}_+; \dot{B}_{\infty, \infty}^1)} \leq C \|u_0\|_{BMO^{-1}} (1 + \|u_0\|_{BMO^{-1}}) \text{ for all } m \in \mathbb{N}.$$

As to the space-time regularity of the *mild* solutions to the NS equations, when $u_0 \in L^n(\mathbb{R}^n)$, in [4], Dong and Du established the result

$$\|t^{\frac{m}{2}+k} \partial_t^k \nabla^m u\|_{L^{n+2}(\mathbb{R}^n \times (0, T^*))} \leq C \text{ for all } k, m \in \mathbb{N},$$

where T^* is the maximum existence time. Vary recently, inspired by the results of [4] and [11], Du in [6] proved that the Koch-Tataru's solution is space-time regularity. More precisely, the author established that there holds

$$t^{\frac{m}{2}+k} \partial_t^k \nabla^m u \in Z_{T^*} \text{ for all } k, m \in \mathbb{N},$$

with the initial data $u_0 \in BMO^{-1}$. On the other hand, with suitable regularities for the solution u to the NS equations, Chemin [1, 2] proved that the existence and uniqueness of the trajectory to u , moreover, this trajectory is Hölder continuous with respect to the space variables.

During the past several decades, there have been many attempts on rigorous mathematical analysis on the nematic liquid crystal flows, see, for example, [7, 8, 13–16, 24–32, 37–40] and the references therein. If $|\nabla d|^2 d$ in (1.2) is replaced by $\frac{(1-|d|^2)d}{\varepsilon}$ (ε is a positive parameter), thus the Dirichlet energy

$$\frac{1}{2} \int_{\mathbb{R}^n} |\nabla d|^2 dx \quad \text{for } d : \mathbb{R}^n \rightarrow \mathbb{S}^2$$

is replaced by the Ginzburg-Landau energy

$$\int_{\mathbb{R}^n} \left(\frac{1}{2} |\nabla d|^2 + \frac{(1-|d|^2)^2}{4\varepsilon^2} \right) dx \quad \text{for } d : \mathbb{R}^n \rightarrow \mathbb{R}^n, \varepsilon > 0.$$

In this case, the system has been studied by a series of papers by Lin [25] and Lin and Liu [27, 28]. More precisely, they proved in [27] the local classical solutions and the global existence of weak solutions in dimensions two and three, and for any fixed ε , they also obtained the existence and uniqueness of global classical solution either in dimension two or dimension three for large fluid viscosity ν . However, as the authors pointed out in [27], it is a challenging problem to study the limiting case as ε tends to zero. Later, in [28], they proved partial regularity of weak solutions in dimension three. Hu and Wang [15] established the existence of global strong solution in suitable regular space and proved that all weak solutions constructed in [27] must be equal to the unique strong solution. Compared with these results, the studies for system (1.1)–(1.4) were only started in recent years. In papers Lin et al. [26] and Hong [14], the authors proved that there exists global Leray-Hopf type weak solutions to (1.1)–(1.4) with suitable boundary condition in dimension two, and established that the solutions are smooth away from at most finitely many singular times which is similar as that for the heat flows of harmonic maps (see [35]). The uniqueness of such weak solutions was subsequently obtained by Lin and Wang [29] and Xu and Zhang [40]. the existence of local-in-time strong solution with large initial value and global-in-time strong solution with small initial value of system (1.1)–(1.4) were also considered by many authors, we refer the readers to see [13, 24, 30, 32, 39] and the references cited therein. Recently, Wang in [38] used the framework of Koch and Tataru [20] to proved that if the initial data $(u_0, d_0) \in BMO^{-1} \times BMO$ with small norm, then system (1.1)–(1.4) exists a global-in time Koch-Tataru type solution. For the regularity issues of solutions to system (1.1)–(1.4), in [7, 8], Du and Wang used the frameworks of the Germain, Pavlović and Staffilani [11] and the Dong and Du [4] to study the rgularity of the Koch-Tataru type solutions to (1.1)–(1.4), and proved that the solution obtained in [38] has arbitrary spatial and temporal regularity

Since the nematic liquid crystal flows (1.1)–(1.4) is a strong coupling system between the incompressible NS equations and the transport heat flow of harmonic maps, there are some similar properties between the NS eqations and the nematic liquid crystal flows. In fact, when researchers studied system (1.1)–(1.4), the solutions to system (1.1)–(1.4) share the similar scaling properties of solutions to the Navier–Stokes equations. That is, if $(u(x, t), d(x, t), P(x, t))$ solves (1.1)–(1.4), then

$$(u_\lambda(x, t), d_\lambda(x, t), P_\lambda(x, t)) := (\lambda u(\lambda x, \lambda^2 t), d(\lambda x, \lambda^2 t), \lambda^2 P(\lambda x, \lambda^2 t))$$

for any $\lambda > 0$ is also a solution of (1.1)–(1.3) with the initial data $(u_{0\lambda}(x), d_{0\lambda}(x)) := (\lambda u_0(\lambda x), d_0(\lambda x))$. These useful properties for the system (1.1)–(1.4) lead to the following definition. A function space (X, Y) called a critical space for (1.1)–(1.4) if it is invariant under the scaling

$$(f_\lambda(x), g_\lambda(x)) := (\lambda f(\lambda x), g(\lambda x)), \text{ for all } (f, g) \in X \times Y.$$

It is easy to varify that the spaces $L^n(\mathbb{R}^n) \times \dot{W}^{1,n}(\mathbb{R}^n)$ and $BMO^{-1} \times BMO$ are critical spaces for system (1.1)–(1.4). We also notice that $BMO^{-1} \times BMO$ may be regard as the largest critical space for initial data, where such well-posedness of system (1.1)–(1.4) can be constructed.

Motivated by the works of Koch and Tataru [20], Germain, Pavlović and Staffilani [11], Dong and Du [4] and Zhang et al. [41] on the NS equations, and by the works of Du and Wang [7, 8], and Wang [38] on the nematic liquid crystal flows, in the present paper, we shall consider the decay estimates for any space-time derivative of the Koch-Tataru type solution involving some borderline Besov space norms. As a corollary, we also present the corresponding decay estimate in time. Though this work is partially enlightened by the paper [41] of Zhang et al., who considered the regularity of Koch-Tataru solution to the NS equations, we have a more sophisticated system to estimate due to the coupling between the velocity field u and the orientation d . Moreover, we overcome the difficulties caused by the operator ∂_t in our proofs. On the other hand, inspired by papers [1, 2] of Chemin, we shall study the trajectories of the Koch-Tataru type solution to (1.1)–(1.4). In order to state our main results, we first recall the definition of the spaces BMO , BMO^{-1} and the existence space of Koch-Tataru type solutions to the nematic liquid crystal flows, for more details about these space, we refer to [7, 8, 20, 22, 38, 41] and the references therein.

Definition 1.1 1. Let W be the solution of $W_t - \Delta W = 0$ with initial data f . Denote

$$[f]_{BMO} := \sup_{0 < r \leq +\infty; x \in \mathbb{R}^n} \left(r^{-n} \int_0^{r^2} \int_{|y-x| < r} |\nabla W|^2 dy dt \right)^{\frac{1}{2}};$$

$$\|f\|_{BMO^{-1}} := \sup_{0 < r \leq +\infty; x \in \mathbb{R}^n} \left(r^{-n} \int_0^{r^2} \int_{|y-x| < r} |W|^2 dy dt \right)^{\frac{1}{2}}.$$

We say a function $f \in L^1_{loc}(\mathbb{R}^n)$ is in $BMO(\mathbb{R}^n)$ if the semi-norm $[f]_{BMO(\mathbb{R}^n)}$ is finite, and we say f is in $BMO^{-1}(\mathbb{R}^n)$ if the norm $\|f\|_{BMO^{-1}}$ is finite. Clearly the divergence of a vector field with components in $BMO(\mathbb{R}^n)$ is in $BMO^{-1}(\mathbb{R}^n)$, i.e., $\nabla \cdot (BMO(\mathbb{R}^n))^n = BMO^{-1}(\mathbb{R}^n)$.

2. We say a function f defined on \mathbb{R}_+^{n+1} belongs to the space $X(\mathbb{R}^n)$ provided

$$\|f\|_X := \sup_{0 < t \leq \infty} \|f(\cdot, t)\|_{L^\infty} + \|f\|_X < +\infty,$$

with

$$\|f\|_X := \sup_{0 < t \leq +\infty} \sqrt{t} \|\nabla f(\cdot, t)\|_{L^\infty} + \sup_{x \in \mathbb{R}^n, 0 < r^2 \leq +\infty} \left(r^{-n} \int_0^{r^2} \int_{|y-x| < r} |\nabla f(y, t)|^2 dy dt \right)^{\frac{1}{2}}.$$

We say a function g defined on \mathbb{R}_+^{n+1} belongs to the space $Z(\mathbb{R}^n)$ provided

$$\|g\|_Z := \sup_{0 < t \leq +\infty} \sqrt{t} \|g(\cdot, t)\|_{L^\infty} + \sup_{x \in \mathbb{R}^n, 0 < r^2 \leq +\infty} \left(r^{-n} \int_0^{r^2} \int_{|y-x| < r} |g(y, t)|^2 dy dt \right)^{\frac{1}{2}} < \infty.$$

It is easy to see that $(X, \|\cdot\|_X)$ and $(Z, \|\cdot\|_Z)$ are two Banach space, and we say $Z \times X$ is the Koch-Tataru existence space for the nematic liquid crystal flow (1.1)–(1.4). We also need to recall the following existence and regularity results for the nematic liquid crystal flows with initial data (u_0, d_0) in $BMO^{-1} \times BMO$, which was proved by Wang [38], and Du and Wang [7, 8].

Theorem 1.2 (see Theorems 1.6 of [38] and Theorem 1.5 of [7, 8]) There exists $\varepsilon = \varepsilon(n) > 0$ small enough, and $C_0 > 0$, such that if $u_0 \in BMO^{-1}$ with $\nabla \cdot u_0 = 0$, and $d_0 \in BMO$ satisfies

$$\|u_0\|_{BMO^{-1}} + [d_0]_{BMO} \leq \varepsilon,$$

then there exists a unique global-in-time solution $(u, d) \in Z \times X$ to the nematic liquid crystal flow (1.1)–(1.4) so that

$$\|u\|_Z + \|d\|_X \leq C_0 \varepsilon.$$

Moreover, for any integers $k, m \geq 0$, there exists a positive constant $C_{k,m}$, such that the unique global-in-time solution (u, d) satisfies

$$\|t^{k+\frac{m}{2}} \partial_t^k \nabla^m u\|_Z + \|t^{k+\frac{m}{2}} \partial_t^k \nabla^m d\|_X \leq C_{k,m} \varepsilon. \quad (1.5)$$

Now we present our main results as follows.

Theorem 1.3 *There exists $\varepsilon = \varepsilon(n) > 0$ small enough, if $u_0 \in BMO^{-1}$ with $\nabla \cdot u_0 = 0$, and $d_0 \in BMO$ satisfies*

$$\|u_0\|_{BMO^{-1}} + [d_0]_{BMO} \leq \varepsilon,$$

then for any integer $k, m \geq 0$, there exists a constant $C_{k,m}$, such that the global-in-time solution (u, d) to the nematic liquid crystal flow (1.1)–(1.4) present by Theorem 1.2 satisfies

$$\|t^{k+\frac{m}{2}} (\partial_t^k \nabla^m u, \partial_t^k \nabla^m \nabla d)\|_{\tilde{L}^\infty(\mathbb{R}_+; \dot{B}_{\infty,\infty}^{-1}) \cap \tilde{L}^1(\mathbb{R}_+; \dot{B}_{\infty,\infty}^1)} \leq C_{m,k} \varepsilon. \quad (1.6)$$

Obviously, Theorem 1.3 implies the following decay estimate immediately:

Corollary 1.4 *Under the assumptions of Theorem 1.3, the global-in-time solution (u, d) to the nematic liquid crystal flow (1.1)–(1.4) present by Theorem 1.2 satisfies*

$$\|(\partial_t^k \nabla^m u, \partial_t^k \nabla^m \nabla d)\|_{\dot{B}_{\infty,\infty}^{-1}} \leq C_{k,m} t^{-\frac{m}{2}-k},$$

for all $t \in \mathbb{R}_+$ and $k, m \geq 0$.

By using Theorem 3.2 from Chemin [1] (or Theorem 3.2.2 from [2]) and the result of the above Theorem 1.3 for the case $k = m = 0$, we have the following result concerning the trajectories of the Koch-Tataru solution to (1.1)–(1.4) present by Theorem 1.2.

Theorem 1.5 *Under the assumptions of Theorem 1.3, there exists a unique continuous mapping γ from \mathbb{R}_+^{n+1} to \mathbb{R}^n such that*

$$\gamma(x, t) = x + \int_0^t (u(\gamma(x, \tau), \tau), \nabla d(\gamma(x, \tau), \tau)) d\tau$$

and

$$|\gamma(x_1, t) - \gamma(x_2, t)| \leq C |x_1 - x_2|^{1-C\varepsilon} \exp(C\varepsilon t) \quad (1.7)$$

for all $t > 0$ and $|x_1 - x_2| \leq 1$.

The remaining parts of the present paper are organized as follows. In section 2, we introduce the Littlewood-Paley decomposition, the definition of Besov space, and some useful estimates of the semi-group of the heat equation on distributions the Fourier transform of which is supported in a rang. In section 3, by using Fourier local analysis method, we give the proof of Theorem 1.3. The last section is devoted to proving Theorem 1.5. Throughout this paper, we use $\|\cdot\|_X$ to denote the norm of the scalar X -functions or the norm of the n -vector X -functions. We also denote by C the various positive constant only depending on the dimension number n , in particular, we denote by $C = C_{a,b,\dots}$ the positive constant which depends only on the dimension number n and the indicated parameters a, b, \dots . All such constants may vary from line to line.

2 Preliminaries

In this section, we are going to recall the dyadic partition of unity in the Fourier variable, the so-called, Littlewood–Paley theory, the definition of Besov space, and some classical results on the heat operator. Part of the materials presented here can be found in [1, 2, 22, 40, 41]. Let $\mathcal{S}(\mathbb{R}^n)$ be the Schwartz class of rapidly decreasing functions. Let $\varphi \in \mathcal{S}(\mathbb{R}^3)$ with values in $[0, 1]$ such that φ is supported in $\mathfrak{C} = \{\xi \in \mathbb{R}^n : \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$ and

$$\sum_{j \in \mathbb{Z}} \varphi_j(\xi) = 1 \quad \text{for any } \xi \in \mathbb{R}^n \setminus \{0\},$$

where $\varphi_j(\xi) \triangleq \varphi(2^{-j}\xi)$. Denoting $h_j = \mathcal{F}^{-1}\varphi_j$, we define the homogeneous dyadic blocks as

$$\Delta_j u \triangleq \varphi(2^{-j}D)u = \int_{\mathbb{R}^n} h_j(y)u(x-y)dy \quad \text{and } S_j f \triangleq \sum_{k \leq j-1} \Delta_k f, \quad \forall j \in \mathbb{Z}, \quad (2.1)$$

where $D = (D_1, D_2, \dots, D_n)$ and $D_j = i^{-1}\partial_{x_j}$, ($j = 1, 2, 3$) with $i^2 = -1$. Let $\mathcal{S}'_h(\mathbb{R}^n)$ be the space of tempered distributions u such that

$$\lim_{\lambda \rightarrow \infty} \|\theta(\lambda D)u\|_{L^\infty} = 0 \quad \text{for any } \theta \in \mathcal{D}(\mathbb{R}^n),$$

where $\mathcal{D}(\mathbb{R}^n)$ is the space of smooth compactly supported functions on \mathbb{R}^n . Then we have the formal decomposition

$$u = \sum_{j \in \mathbb{Z}} \Delta_j u, \quad \forall u \in \mathcal{S}'_h(\mathbb{R}^n).$$

Informally, $\Delta_j = S_{j+1} - S_j$ is a frequency projection to the annulus $\{|\xi| \approx 2^j\}$, while S_j is the frequency projection to $\{0 < |\xi| \lesssim 2^j\}$. One easily verifies that with the above choice of φ ,

$$\Delta_j \Delta_k f \equiv 0 \text{ if } |j - k| \geq 2 \text{ and } \Delta_j (S_{k-1} f \Delta_k f) \equiv 0 \text{ if } |j - k| \geq 5.$$

We recall now the definition of the homogeneous Besov spaces from [2, 22].

Definition 2.1 (homogeneous Besov spaces $\dot{B}_{p,r}^s(\mathbb{R}^n)$) Let $s \in \mathbb{R}$, $1 \leq p, r \leq \infty$, we set

$$\|u\|_{\dot{B}_{p,r}^s} \triangleq \begin{cases} \left(\sum_{j \in \mathbb{Z}} 2^{jsr} \|\Delta_j u\|_{L^p}^r \right)^{\frac{1}{r}} & \text{for } 1 \leq r < \infty, \\ \sup_{j \in \mathbb{Z}} 2^{js} \|\Delta_j u\|_{L^p} & \text{for } r = \infty. \end{cases}$$

- For $s < \frac{n}{p}$ (or $s = \frac{n}{p}$ if $r = 1$), we define $\dot{B}_{p,r}^s(\mathbb{R}^n) \triangleq \{u \in \mathcal{S}'_h(\mathbb{R}^n) \mid \|u\|_{\dot{B}_{p,r}^s} < \infty\}$.
- If $k \in \mathbb{N}$ and $\frac{n}{p} + k \leq s < \frac{n}{p} + k + 1$ (or $s = \frac{n}{p} + k + 1$ if $r = 1$), the $\dot{B}_{p,r}^s(\mathbb{R}^n)$ is defined as the subset of distributions $u \in \mathcal{S}'_h(\mathbb{R}^n)$ such that $\partial_x^\gamma u \in \dot{B}_{p,r}^{s-k}(\mathbb{R}^n)$ whenever $\partial_x^\gamma := \partial_{x_1}^{\gamma_1} \partial_{x_2}^{\gamma_2} \cdots \partial_{x_n}^{\gamma_n}$ with $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_n)$ and $|\gamma| = k$. For short, we just denote the space $\dot{B}_{p,r}^s(\mathbb{R}^n)$ by $\dot{B}_{p,r}^s$.

We still need to define the so called Chemin-Lerner type spaces $\tilde{L}_T^\rho(\dot{B}_{p,r}^s(\mathbb{R}^n))$, i.e.,

Definition 2.2 Let $s \in \mathbb{R}$, $1 \leq p, r, \rho \leq \infty$ and $T \in (0, +\infty]$, we define $\tilde{L}_T^\rho(\dot{B}_{p,r}^s(\mathbb{R}^n))$ as the completion of $C([0, T]; \mathcal{S}(\mathbb{R}^3))$ by the norm

$$\|f\|_{\tilde{L}_T^\rho(\dot{B}_{p,r}^s)} \triangleq \left(\sum_{j \in \mathbb{Z}} 2^{jsr} \|\Delta_j f\|_{L_T^\rho(L^p)}^r \right)^{\frac{1}{r}},$$

with the usual change if $r = \infty$. For short, we just denote this space by $\tilde{L}_T^\rho(\dot{B}_{p,r}^s)$.

Remark 2.3 By virtue of the Minkowski inequality, one obtains

$$\|u\|_{\tilde{L}_T^\rho(\dot{B}_{p,r}^s)} \leq \|u\|_{L_T^\rho(\dot{B}_{p,r}^s)} \text{ if } r \geq \rho, \text{ and } \|u\|_{\tilde{L}_T^\rho(\dot{B}_{p,r}^s)} \geq \|u\|_{L_T^\rho(\dot{B}_{p,r}^s)} \text{ if } r \leq \rho.$$

The following lemma gives the way the product acts on Chemin-Lerner type spaces.

Lemma 2.4 Let $s_1, s_2 \in \mathbb{R}$, $p, p_1, p_2, q, q_1, q_2, r_1, r_2 \in [1, \infty]$ such that

$$\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q}, \frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{r} \leq 1 \text{ and } p \geq \max\{p_1, p_2\}$$

If $s_1 + s_2 > 0$, the product maps $\tilde{L}_T^{q_1}(\dot{B}_{p_1,r_1}^{s_1}) \times \tilde{L}_T^{q_2}(\dot{B}_{p_2,r_2}^{s_2})$ into $\tilde{L}_T^q(\dot{B}_{p,r}^{s_1+s_2-n(\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p})})$. If $s_1 = -s_2$ and $r = 1$, the product maps $\tilde{L}_T^{q_1}(\dot{B}_{p_1,r_1}^{s_1}) \times \tilde{L}_T^{q_2}(\dot{B}_{p_2,r_2}^{s_2})$ into $\tilde{L}_T^q(\dot{B}_{p,\infty}^{-n(\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p})})$.

We end the section by the following lemma, which gives useful estimates for the semi-group of the heat equation restricted to functions with compact supports away from the origin in Fourier transform variables (see Lemma 2.1 in [1], Lemma 2.1.2 in [2], and Lemmas 2.1 and 3.1 in [41]).

Lemma 2.5 Let φ be the smooth function defined as the beginning of this section, and $\tilde{\varphi}$ be the smooth function supported in the ring $\tilde{\mathcal{C}} = \{\xi \in \mathbb{R}^n; \frac{3}{8} \leq |\xi| \leq \frac{10}{3}\}$ such that $\tilde{\varphi}\varphi = \varphi$. Let $1 \leq i, j, k \leq n$, $m \geq 0$

$$\begin{aligned} g_q^{i,j,k}(x, t) &:= \int_{\mathbb{R}^n} e^{ix \cdot \xi} \varphi(2^{-q}\xi) e^{-t|\xi|^2} \left(\delta_{ij} - \frac{\xi_i \xi_j}{|\xi|^2} \right) \xi_k d\xi, \\ g_{1,q}^{i,j}(x, t) &:= \int_{\mathbb{R}^n} e^{ix \cdot \xi} \varphi(2^{-q}\xi) e^{-t|\xi|^2} \left(\delta_{ij} - \frac{\xi_i \xi_j}{|\xi|^2} \right) d\xi, \\ g_{2,q}^{i,j}(x, t) &:= \int_{\mathbb{R}^n} e^{ix \cdot \xi} \varphi(2^{-q}\xi) e^{-t|\xi|^2} d\xi \end{aligned}$$

and

$$g_{3,q}(x, t) := \int_{\mathbb{R}^n} e^{ix \cdot \xi} \tilde{\varphi}(2^{-q}\xi) t^{\frac{m}{2}} \xi^\gamma e^{-t|\xi|^2} d\xi,$$

where $|\gamma| = m$. Then there exist three positive constants c , C and C_m such that, for all $q \in \mathbb{Z}$ and $t > 0$, we have

$$\begin{aligned} |g_q^{i,j,k}(x, t)| &\leq C \frac{2^{q(n+1)}}{1 + |2^q x|^{2n}} e^{-ct2^{2q}}; \\ |t^{\frac{m}{2}} \nabla^m g_q^{i,j,k}(x, t)| &\leq C_m \frac{2^{q(n+1)}}{1 + |2^q x|^{2n}} e^{-ct2^{2q}}; \\ |g_{1,q}^{i,j}(x, t)| + |g_{2,q}(x, t)| &\leq C \frac{2^{qn}}{1 + |2^q x|^{2n}} e^{-ct2^{2q}}; \\ |t^{\frac{m}{2}} \nabla^m g_{2,q}(x, t)| + |g_{3,q}(x, t)| &\leq C_m \frac{2^{qn}}{1 + |2^q x|^{2n}} e^{-ct2^{2q}}. \end{aligned}$$

3 The proof of Theorem 1.3

In this section, we shall give the proof of Theorem 1.3. We first give the following lemma, which gives some estimates for the linear heat equations.

Lemma 3.1 Taking a constant $\varepsilon = \varepsilon(n) > 0$ small enough, and the initial data $(u_0, d_0) \in BMO^{-1} \times BMO$ with $\|u_0\|_{BMO^{-1}} + \|d_0\|_{BMO} \leq \varepsilon$, then for any integer $m \geq 0$, we have

$$\|t^{\frac{m}{2}} (\nabla^m e^{t\Delta} u_0, \nabla^{m+1} e^{t\Delta} d_0)\|_{\tilde{L}^\infty(\mathbb{R}_+; \dot{B}_{\infty,\infty}^{-1}) \cap \tilde{L}^1(\mathbb{R}_+; \dot{B}_{\infty,\infty}^1)} \leq C_m \varepsilon. \quad (3.1)$$

Proof. At the beginning, we need to recall Lemma 2.2 in [7], Du and Wang proved that under the assumptions of Lemma 3.1, there holds

$$\|t^{\frac{m}{2}} \nabla^m e^{t\Delta} u_0\|_Z \leq C_m \|u_0\|_{BMO^{-1}} \quad \text{and} \quad \|t^{\frac{m}{2}} \nabla^m e^{t\Delta} d_0\|_X \leq C_m [d_0]_{BMO}, \quad (3.2)$$

where the norms $\|\cdot\|_Z$ and $\|\cdot\|_X$ defined as Definition 1.1.

In what follows, we are in a position to the proof of (3.1). Firstly, we prove that

$$\|t^{\frac{m}{2}} \nabla^m e^{t\Delta} u_0\|_{\tilde{L}^\infty(\mathbb{R}_+; \dot{B}_{\infty,\infty}^{-1}) \cap \tilde{L}^1(\mathbb{R}_+; \dot{B}_{\infty,\infty}^1)} \leq C_m \|u_0\|_{BMO^{-1}}. \quad (3.3)$$

Notice that for all $t \in [2^{-2q}, \infty)$, it follows that

$$2^{-q} \|\Delta_q t^{\frac{m}{2}} \nabla^m e^{t\Delta} u_0\|_{L^\infty} \leq 2^{-q} \|t^{\frac{m}{2}} \nabla^m e^{t\Delta} u_0\|_{L^\infty} \leq 2^{-q} t^{-\frac{1}{2}} \|t^{\frac{m}{2}} \nabla^m e^{t\Delta} u_0\|_Z \leq C_m \|u_0\|_{BMO^{-1}}, \quad (3.4)$$

where we have used (3.2) in the last inequality. For $t \in (0, 2^{-2q})$, by using the embedding $BMO^{-1} \hookrightarrow \dot{B}_{\infty,\infty}^{-1}$ and Lemma 2.5, we have

$$\begin{aligned} 2^{-q} \|\Delta_q t^{\frac{m}{2}} \nabla^m e^{t\Delta} u_0\|_{L^\infty} &= 2^{-q} \|g_{3,q} * \Delta_q u_0\|_{L^\infty} \leq C_m 2^{-q} \|\Delta_q u_0\|_{L^\infty} \\ &\leq C_m \|u_0\|_{\dot{B}_{\infty,\infty}^{-1}} \leq C_m \|u_0\|_{BMO^{-1}}. \end{aligned} \quad (3.5)$$

On the other hand, again by $BMO^{-1} \hookrightarrow \dot{B}_{\infty,\infty}^{-1}$ and Lemma 2.5, we get

$$\begin{aligned} 2^q \|\Delta_q t^{\frac{m}{2}} \nabla^m e^{t\Delta} u_0\|_{L^1(\mathbb{R}_+; L^\infty)} &= 2^q \int_0^\infty \|g_{3,q} * \Delta_q u_0\|_{L^\infty} dt \leq C_m 2^q \int_0^\infty e^{-ct2^{2q}} dt \|\Delta_q u_0\|_{L^\infty} \\ &\leq C_m 2^{-q} \|\Delta_q u_0\|_{L^\infty} \leq C_m \|u_0\|_{\dot{B}_{\infty,\infty}^{-1}} \leq C_m \|u_0\|_{BMO^{-1}}, \end{aligned}$$

which together with (3.4) and (3.5) implies that (3.3). In a similar way, we can prove the rest part of (3.2). In fact, for all $t \in [2^{-2q}, \infty)$, we have

$$2^{-q} \|\Delta_q t^{\frac{m}{2}} \nabla^{m+1} e^{t\Delta} d_0\|_{L^\infty} \leq 2^{-q} \|\nabla(t^{\frac{m}{2}} \nabla^m e^{t\Delta} d_0)\|_{L^\infty} \leq 2^{-q} t^{-\frac{1}{2}} \|t^{\frac{m}{2}} \nabla^m e^{t\Delta} d_0\|_X \leq C_m [d_0]_{BMO}, \quad (3.6)$$

and for $t \in (0, 2^{-2q})$, by using the embedding $BMO \hookrightarrow \dot{B}_{\infty,\infty}^0$ and Lemma 2.5, we have

$$\begin{aligned} 2^{-q} \|\Delta_q t^{\frac{m}{2}} \nabla^{m+1} e^{t\Delta} d_0\|_{L^\infty} &= 2^{-q} \|g_{3,q} * \Delta_q \nabla d_0\|_{L^\infty} \leq C_m 2^{-q} \|\Delta_q \nabla d_0\|_{L^\infty} \\ &\leq C_m \|\Delta_q d_0\|_{L^\infty} \leq C_m \|d_0\|_{\dot{B}_{\infty,\infty}^0} \leq C_m [d_0]_{BMO}. \end{aligned} \quad (3.7)$$

Again by $BMO \hookrightarrow \dot{B}_{\infty,\infty}^0$ and Lemma 2.5, we get

$$\begin{aligned} 2^q \|\Delta_q t^{\frac{m}{2}} \nabla^{m+1} e^{t\Delta} d_0\|_{L^1(\mathbb{R}_+; L^\infty)} &= 2^q \int_0^\infty \|g_{3,q} * \Delta_q \nabla d_0\|_{L^\infty} dt \leq C_m 2^q \int_0^\infty e^{-ct2^{2q}} dt \|\Delta_q \nabla d_0\|_{L^\infty} \\ &\leq C_m 2^{-q} \|\Delta_q \nabla d_0\|_{L^\infty} \leq C_m \|d_0\|_{\dot{B}_{\infty,\infty}^0} \leq C_m [d_0]_{BMO}, \end{aligned} \quad (3.8)$$

Combining (3.6), (3.7) and (3.8) together, it follows that

$$\|t^{\frac{m}{2}} \nabla^{m+1} e^{t\Delta} d_0\|_{\tilde{L}^\infty(\mathbb{R}_+; \dot{B}_{\infty,\infty}^{-1}) \cap \tilde{L}^1(\mathbb{R}_+; \dot{B}_{\infty,\infty}^1)} \leq C_m [d_0]_{BMO},$$

which together with (3.3) verifies (3.1). This completes the proof Lemma 3.1. \square

We are now in a position to the proof of Theorem 1.3. We first rewrite the system (1.1)–(1.4) as an integral system:

$$\begin{cases} u := e^{-t\Delta} u_0 + \mathbb{T}_1(u, d)(x, t), \\ d := e^{-t\Delta} d_0 + \mathbb{T}_2(u, d)(x, t), \end{cases} \quad (3.9)$$

with

$$\begin{cases} \mathbb{T}_1(u, d)(x, t) := - \int_0^t e^{-(t-\tau)\Delta} \mathbb{P} \nabla \cdot (u \otimes u + \nabla d \odot \nabla d)(\cdot, \tau) d\tau, \\ \mathbb{T}_2(u, d)(x, t) := \int_0^t e^{-(t-\tau)\Delta} (|\nabla d|^2 d - u \cdot \nabla d)(\cdot, \tau) d\tau, \end{cases}$$

where $\mathbb{P} := I + \nabla(-\Delta)^{-1}\text{div}$ is the Helmholtz-Weyl projection operator which has the matrix symbol with components

$$(\widehat{\mathbb{P}}(\xi))_{j,k} = \delta_{jk} - \xi_j \xi_k |\xi|^{-2} \quad \text{with } j, k = 1, 2, \dots, n,$$

where δ_{jk} is Kronecker symbol, and \otimes denotes tensor product.

In what follows, we shall divided the proof of the Theorem 1.3 into two steps.

Step 1. Estimate (1.6) with the case of $k = 0$ and $m \geq 0$, i.e., we shall prove

$$\|t^{\frac{m}{2}}(\nabla^m u, \nabla^{m+1} d)\|_{\tilde{L}^\infty(\mathbb{R}_+; \dot{B}_{\infty, \infty}^{-1}) \cap \tilde{L}^1(\mathbb{R}_+; \dot{B}_{\infty, \infty}^1)} \leq C_m \varepsilon.$$

By using Lemma 3.1, to prove the above inequality, it sufficient to verify for any positive integers $m \geq 0$,

$$\|t^{\frac{m}{2}}(\nabla^m \mathbb{T}_1(u, d), \nabla^{m+1} \mathbb{T}_2(u, d))(x, t)\|_{\tilde{L}^\infty(\mathbb{R}_+; \dot{B}_{\infty, \infty}^{-1}) \cap \tilde{L}^1(\mathbb{R}_+; \dot{B}_{\infty, \infty}^1)} \leq C_m \varepsilon. \quad (3.10)$$

We first give the estimate of $\mathbb{T}_1(u, d)$. For $t \in [2^{-2q}, \infty)$, by using (1.5) and Lemma 2.5, we have

$$\begin{aligned} 2^{-q} \|\Delta_q t^{\frac{m}{2}} \nabla^m \mathbb{T}_1(u, d)(\cdot, t)\|_{L^\infty} &\leq 2^{-q} \|t^{\frac{m}{2}} \nabla^m \mathbb{T}_1(u, d)(\cdot, t)\|_{L^\infty} \\ &\leq 2^{-q} \|t^{\frac{m}{2}} \nabla^m u(\cdot, t)\|_{L^\infty} + 2^{-q} \|t^{\frac{m}{2}} \nabla^m e^t \Delta u_0\|_{L^\infty} \\ &\leq 2^{-q} t^{-\frac{1}{2}} \|t^{\frac{m}{2}} \nabla^m u\|_Z + C_m \|u_0\|_{BMO^{-1}} \\ &\leq C_m (\varepsilon + \|u_0\|_{BMO^{-1}}) \leq C_m \varepsilon, \end{aligned} \quad (3.11)$$

where we have used (3.9) in the inequalities above. For $t \in (0, 2^{-2q})$, by denoting

$$\mathbf{F}(x, t) := (u \otimes u + \nabla d \odot \nabla d)(x, t),$$

we can rewrite $\Delta_q \mathbb{T}_1(u, d)$ for any $q \in \mathbb{Z}$ as

$$\begin{aligned} \Delta_q \mathbb{T}_1(u, d)(x, t) &= \int_0^{\frac{t}{2}} \int_{|y| \geq 2^{1-q}} g_q(y, t - \tau) \cdot \mathbf{F}(x - y, \tau) dy d\tau \\ &\quad + \int_{\frac{t}{2}}^t \int_{|y| \geq 2^{1-q}} g_q(y, t - \tau) \cdot \mathbf{F}(x - y, \tau) dy d\tau \\ &\quad + \int_0^{\frac{t}{2}} \int_{|y| \leq 2^{1-q}} g_q(y, t - \tau) \cdot \mathbf{F}(x - y, \tau) dy d\tau \\ &\quad + \int_{\frac{t}{2}}^t \int_{|y| \leq 2^{1-q}} g_q(y, t - \tau) \cdot \mathbf{F}(x - y, \tau) dy d\tau \\ &:= (F_{1,q} + F_{2,q} + F_{3,q} + F_{4,q})(x, t), \end{aligned}$$

where $(g_q \cdot \mathbf{F})^i = g_q^{i,j,k} \mathbf{F}^{j,k}$ and $g_q^{i,j,k}$ are given by Lemma 2.5. We shall estimate term by term from $F_{1,q}$ to $F_{4,q}$. Indeed thanks to (1.5) and Lemma 2.5, we have

$$\begin{aligned} |t^{\frac{m}{2}} \nabla^m F_{1,q}(x, t)| &= \int_0^{\frac{t}{2}} \int_{|y| \geq 2^{1-q}} \left(\frac{t}{t-\tau}\right)^{\frac{m}{2}} (t-\tau)^{\frac{m}{2}} (\nabla^m g_q)(y, t-\tau) \cdot \mathbf{F}(x-y, \tau) dy d\tau \\ &\leq C_m \int_0^{\frac{t}{2}} \int_{|y| \geq 2^{1-q}} \frac{2^{q(n+1)}}{1 + (2^q |y|)^{2n}} e^{-c(t-\tau)2^{2q}} |\mathbf{F}(x-y, \tau)| dy d\tau \\ &\leq C_m \sum_{\ell \in \mathbb{Z}^n \setminus \{0\}} \int_0^t \int_{y \in 2^{-q}(\ell + [0,1]^n)} \frac{2^{q(n+1)}}{1 + (2^q |y|)^{2n}} |\mathbf{F}(x-y, \tau)| dy d\tau \\ &\leq C_m \sum_{\ell \in \mathbb{Z}^n \setminus \{0\}} \frac{2^{q(n+1)}}{|\ell|^{2n}} \int_0^{2^{-2q}} \int_{y \in 2^{-q}(\ell + [0,1]^n)} (|u(x-y, \tau)|^2 + |\nabla d(x-y, \tau)|^2) dy d\tau \\ &\leq C_m \sum_{\ell \in \mathbb{Z}^n \setminus \{0\}} \frac{2^{q(n+1)}}{|\ell|^{2n}} 2^{-qn} (\|u\|_Z^2 + \|d\|_X^2) \leq C_m 2^q (\|u\|_Z^2 + \|d\|_X^2) \leq C_m 2^q \varepsilon^2. \end{aligned}$$

Exactly following the same line, we have

$$\begin{aligned}
|t^{\frac{m}{2}} \nabla^m F_{2,q}(x, t)| &= \int_{\frac{t}{2}}^t \int_{|y| \geq 2^{1-q}} g_q(y, t - \tau) \cdot t^{\frac{m}{2}} \nabla^m \mathbf{F}(x - y, \tau) dy d\tau \\
&\leq C_m \int_{\frac{t}{2}}^t \int_{|y| \geq 2^{1-q}} \frac{2^{q(n+1)}}{1 + (2^q y)^{2n}} e^{-c(t-\tau)2^{2q}} \sum_{i=0}^m \left(|t^{\frac{i}{2}} \nabla^i u(x - y, \tau)|^2 + |t^{\frac{i}{2}} \nabla^i \nabla d(x - y, \tau)|^2 \right) dy d\tau \\
&\leq C_m \sum_{\ell \in \mathbb{Z}^n \setminus \{0\}} \int_{\frac{t}{2}}^t \int_{y \in 2^{-q}(\ell + [0, 1]^n)} \frac{2^{q(n+1)}}{1 + (2^q y)^{2n}} \sum_{i=0}^m \left(|\tau^{\frac{i}{2}} \nabla^i u(x - y, \tau)|^2 + |\tau^{\frac{i}{2}} \nabla^i \nabla d(x - y, \tau)|^2 \right) dy d\tau \\
&\leq C_m \sum_{\ell \in \mathbb{Z}^n \setminus \{0\}} \frac{2^{q(n+1)}}{|\ell|^{2n}} 2^{-qn} \sum_{i=1}^m \left(\|\tau^{\frac{i}{2}} \nabla^i u\|_Z^2 + \|\tau^{\frac{i}{2}} \nabla^i d\|_X^2 \right) \leq C_m 2^q \varepsilon^2; \\
|t^{\frac{m}{2}} \nabla^m F_{3,q}(x, t)| &= \int_0^{\frac{t}{2}} \int_{|y| \leq 2^{1-q}} \left(\frac{t}{t - \tau} \right)^{\frac{m}{2}} (t - \tau)^{\frac{m}{2}} (\nabla^m g_q)(y, t - \tau) \cdot \mathbf{F}(x - y, \tau) dy d\tau \\
&\leq C_m \int_0^{\frac{t}{2}} \int_{|y| \leq 2^{1-q}} \frac{2^{q(n+1)}}{1 + (2^q |y|)^{2n}} e^{-c(t-\tau)2^{2q}} |\mathbf{F}(x - y, \tau)| dy d\tau \\
&\leq C_m 2^{q(n+1)} \int_0^{\frac{t}{2}} \int_{|z-x| \leq 2^{1-q}} |\mathbf{F}(z, \tau)| dz d\tau \\
&\leq C_m 2^{q(n+1)} \int_0^{2^{-2q}} \int_{|z-x| \leq 2^{1-q}} (|u(z, \tau)|^2 + |\nabla d(z, \tau)|^2) dz d\tau \\
&\leq C_m 2^{q(n+1)} 2^{-qn} (\|u\|_Z^2 + \|d\|_X^2) \leq C_m 2^q \varepsilon^2.
\end{aligned}$$

and

$$\begin{aligned}
|t^{\frac{m}{2}} \nabla^m F_{4,q}(x, t)| &= \int_{\frac{t}{2}}^t \int_{|y| \leq 2^{1-q}} g_q(y, t - \tau) \cdot t^{\frac{m}{2}} \nabla^m \mathbf{F}(x - y, \tau) dy d\tau \\
&\leq C_m \int_{\frac{t}{2}}^t \int_{|y| \leq 2^{1-q}} \frac{2^{q(n+1)}}{1 + (2^q |y|)^{2n}} e^{-c(t-\tau)2^{2q}} \sum_{i=0}^m \left(|t^{\frac{i}{2}} \nabla^i u(x - y, \tau)|^2 + |t^{\frac{i}{2}} \nabla^i \nabla d(x - y, \tau)|^2 \right) dy d\tau \\
&\leq C_m \int_{\frac{t}{2}}^t \int_{|y| \leq 2^{1-q}} \frac{2^{q(n+1)}}{1 + (2^q |y|)^{2n}} e^{-c(t-\tau)2^{2q}} \sum_{i=0}^m \left(|\tau^{\frac{i}{2}} \nabla^i u(x - y, \tau)|^2 + |\tau^{\frac{i}{2}} \nabla^i \nabla d(x - y, \tau)|^2 \right) dy d\tau \\
&\leq C_m 2^{q(n+1)} \sum_{i=0}^m \int_0^{\frac{t}{2}} \int_{|z-x| \leq 2^{1-q}} \left(|\tau^{\frac{i}{2}} \nabla^i u(z, \tau)|^2 + |\tau^{\frac{i}{2}} \nabla^i \nabla d(z, \tau)|^2 \right) dz d\tau \\
&\leq C_m 2^{q(n+1)} \sum_{i=0}^m \int_0^{2^{-2q}} \int_{|z-x| \leq 2^{1-q}} \left(|\tau^{\frac{i}{2}} \nabla^i u(z, \tau)|^2 + |\tau^{\frac{i}{2}} \nabla^i \nabla d(z, \tau)|^2 \right) dz d\tau \\
&\leq C_m 2^{q(n+1)} 2^{-qn} \sum_{i=0}^m \left(\|\tau^{\frac{i}{2}} \nabla^i u\|_Z^2 + \|\tau^{\frac{i}{2}} \nabla^i d\|_X^2 \right) \leq C_m 2^q \varepsilon^2.
\end{aligned}$$

As a consequence, if we select $\varepsilon > 0$ small enough, we have

$$|t^{\frac{m}{2}} \nabla^m \Delta_q \mathbb{T}_1(u, d)(x, t)| \leq C_m 2^q \varepsilon^2 \leq C_m 2^q \varepsilon \quad \text{for } t \in (0, 2^{-q}),$$

which together with (3.11) implies that

$$\|t^{\frac{m}{2}} \nabla^m \Delta_q \mathbb{T}_1(u, d)\|_{\tilde{L}^\infty(\mathbb{R}_+; \dot{B}_{\infty, \infty}^{-1})} \leq C_m \varepsilon. \quad (3.12)$$

Now let us turn to estimate the $\tilde{L}^1(\mathbb{R}_+; \dot{B}_{\infty, \infty}^1)$ -norm of $t^{\frac{m}{2}} \nabla^m \mathbb{T}_1(u, d)$, by using (3.9) and Lemma 2.5 again, we have

$$\begin{aligned}
&2^q \int_0^{2^{-2q}} \|t^{\frac{m}{2}} \nabla^m \Delta_q \mathbb{T}_1(u, d)(\cdot, t)\|_{L^\infty} dt \\
&\leq 2^q \int_0^{2^{-2q}} \|t^{\frac{m}{2}} \nabla^m u(\cdot, t)\|_{L^\infty} dt + 2^q \int_0^{2^{-2q}} \|t^{\frac{m}{2}} \nabla^m e^{t\Delta} u_0\|_{L^\infty} dt
\end{aligned}$$

$$\leq 2^q \int_0^{2^{-2q}} t^{-\frac{1}{2}} dt \|t^{\frac{m}{2}} \nabla^m u\|_Z + C_m \|u_0\|_{BMO^{-1}} \leq C_m \varepsilon, \quad (3.13)$$

where we have used (1.5) in the last above inequality. To complete the proof, we still need to estimate the term $2^q \int_{2^{-2q}}^\infty \|t^{\frac{m}{2}} \nabla^m \Delta_q \mathbb{T}_1(u, d)(\cdot, t)\|_{L^\infty} dt$. In order to do it, for all $q \in \mathbb{Z}$, we split $\Delta_q \mathbb{T}_1(u, d)$ as

$$\begin{aligned} \Delta_q \mathbb{T}_1(u, d)(x, t) &= \int_0^{\frac{t}{2}} \Delta_q e^{(t-\tau)\Delta} \mathbb{P} \nabla \cdot \mathbf{F}(x, \tau) d\tau + \int_{\frac{t}{2}}^t \int_{|y| \geq 2t^{\frac{1}{2}}} g_q(y, t-\tau) \cdot \mathbf{F}(x-y, \tau) dy d\tau \\ &\quad + \int_{\frac{t}{2}}^t \int_{|y| \leq 2t^{\frac{1}{2}}} g_{1,q}(y, t-\tau) \nabla \cdot \mathbf{F}(x-y, \tau) dy d\tau \\ &:= (\tilde{F}_{1,q} + \tilde{F}_{2,q} + \tilde{F}_{3,q})(x, t). \end{aligned}$$

By applying (3.9) and Lemma 2.5 that

$$\begin{aligned} &2^q \int_{2^{-2q}}^\infty \|t^{\frac{m}{2}} \nabla^m \tilde{F}_{1,q}(\cdot, t)\|_{L^\infty} dt \\ &= 2^q \int_{2^{-2q}}^\infty \left\| t^{\frac{m}{2}} \nabla^m \Delta_q e^{\frac{t}{2}\Delta} \left(u(x, \frac{t}{2}) - (e^{\frac{t}{2}\Delta} u_0)(x) \right) \right\|_{L^\infty} dt \\ &\leq 2^q \int_{2^{-2q}}^\infty \left(\left\| t^{\frac{m}{2}} \nabla^m \Delta_q e^{\frac{t}{2}\Delta} u(x, \frac{t}{2}) \right\|_{L^\infty} + \left\| t^{\frac{m}{2}} \nabla^m \Delta_q (e^{\frac{t}{2}\Delta} u_0)(x) \right\|_{L^\infty} \right) dt \\ &\leq C_m 2^q \int_{2^{-2q}}^\infty \left(\left\| \int_{\mathbb{R}^n} g_{3,q}(y, \frac{t}{2}) \cdot u(x-y, \frac{t}{2}) dy \right\|_{L^\infty} + \|g_{3,q} * \Delta_q u_0\|_{L^\infty} \right) dt \\ &\leq C_m 2^q \int_{2^{-2q}}^\infty \int_{\mathbb{R}^n} \frac{2^{qn}}{1 + (2^q y)^{2n}} e^{-ct2^{2q}} dy \left(\left\| u(\cdot, \frac{t}{2}) \right\|_{L^\infty} + \|\Delta_q u_0\|_{L^\infty} \right) dt \\ &\leq C_m 2^q \int_{2^{-2q}}^\infty e^{-ct2^{2q}} t^{-\frac{1}{2}} dt \|u\|_Z + C_m 2^q \int_{2^{-2q}}^\infty e^{-ct2^{2q}} dt \|\Delta_q u_0\|_{L^\infty} \\ &\leq C_m \|u\|_Z + C_m 2^{-q} \|\Delta_q u_0\|_{L^\infty} \leq C_m (\|u\|_Z + \|u_0\|_{\dot{B}_{\infty,\infty}^{-1}}) \\ &\leq C_m \varepsilon + C_m \|u_0\|_{BMO^{-1}} \leq C_m \varepsilon, \end{aligned}$$

where we have used the estimate (1.5) and the embedding $BMO^{-1} \hookrightarrow \dot{B}_{\infty,\infty}^{-1}$. Exactly following the same line, we have

$$\begin{aligned} &2^q \int_{2^{-2q}}^\infty \|t^{\frac{m}{2}} \nabla^m \tilde{F}_{2,q}(\cdot, t)\|_{L^\infty} dt \\ &\leq C_m 2^q \int_{2^{-2q}}^\infty \left\| \int_{\frac{t}{2}}^t \int_{|y| \geq 2t^{\frac{1}{2}}} \frac{2^{q(n+1)}}{1 + (2^q |y|)^{2n}} e^{-c(t-\tau)2^{2q}} |t^{\frac{m}{2}} \nabla^m \mathbf{F}(x-y, \tau)| dy d\tau \right\|_{L^\infty} dt \\ &\leq C_m 2^q \int_{2^{-2q}}^\infty \left\| \int_{\frac{t}{2}}^t \int_{|y| \geq 2t^{\frac{1}{2}}} \frac{2^{q(n+1)}}{1 + (2^q |y|)^n} e^{-c(t-\tau)2^{2q}} \sum_{i=0}^m \left(|t^{\frac{i}{2}} \nabla^i u(x-y, \tau)|^2 + |t^{\frac{i}{2}} \nabla^i \nabla d(x-y, \tau)|^2 \right) dy d\tau \right\|_{L^\infty} dt \\ &\leq C_m 2^q \sum_{\ell \in \mathbb{Z}^n \setminus \{0\}} \int_{2^{-2q}}^\infty \left\| \int_{\frac{t}{2}}^t \int_{|y| \in \sqrt{t}(\ell + [0,1]^n)} \frac{2^{q(n+1)}}{1 + (2^q |y|)^{2n}} \sum_{i=0}^m \left(|\tau^{\frac{i}{2}} \nabla^i u(x-y, \tau)|^2 + |\tau^{\frac{i}{2}} \nabla^i \nabla d(x-y, \tau)|^2 \right) dy d\tau \right\|_{L^\infty} dt \\ &\leq C_m 2^q \sum_{\ell \in \mathbb{Z}^n \setminus \{0\}} \int_{2^{-2q}}^\infty \left\| \frac{2^{q(1-n)}}{(|\ell| \sqrt{t})^{2n}} \int_{\frac{t}{2}}^t \int_{|y| \in \sqrt{t}(\ell + [0,1]^{2n})} \sum_{i=0}^m \left(|\tau^{\frac{i}{2}} \nabla^i u(x-y, \tau)|^2 + |\tau^{\frac{i}{2}} \nabla^i \nabla d(x-y, \tau)|^2 \right) dy d\tau \right\|_{L^\infty} dt \\ &\leq C_m 2^q \sum_{\ell \in \mathbb{Z}^n \setminus \{0\}} \int_{2^{-2q}}^\infty \left(\frac{2^{q(1-n)}}{(|\ell| \sqrt{t})^{2n}} t^{\frac{n}{2}} \sum_{i=0}^m \left(\|\tau^{\frac{i}{2}} \nabla^i u\|_Z^2 + \|\tau^{\frac{i}{2}} \nabla^i d\|_X^2 \right) \right) dt \\ &\leq C_m \sum_{\ell \in \mathbb{Z}^n \setminus \{0\}} \frac{1}{|\ell|^{2n}} \sum_{i=0}^m \left(\|\tau^{\frac{i}{2}} \nabla^i u\|_Z^2 + \|\tau^{\frac{i}{2}} \nabla^i d\|_X^2 \right) \leq C_m \varepsilon^2, \end{aligned}$$

and

$$2^q \int_{2^{-2q}}^\infty \|t^{\frac{m}{2}} \nabla^m \tilde{F}_{3,q}(\cdot, t)\|_{L^\infty} dt$$

$$\begin{aligned}
&\leq C_m 2^q \int_{2^{-2q}}^{\infty} \left\| \int_{\frac{t}{2}}^t \int_{|y| \leq 2t^{\frac{1}{2}}} \frac{2^{qn}}{1 + (2^q |y|)^{2n}} e^{-c(t-\tau)2^{2q}} |t^{\frac{m}{2}} \nabla^{m+1} \mathbf{F}(x-y, \tau)| dy d\tau \right\|_{L^\infty} dt \\
&\leq C_m 2^q \int_{2^{-2q}}^{\infty} \left\| \int_{\frac{t}{2}}^t \int_{|y| \leq 2t^{\frac{1}{2}}} \frac{2^{qn}}{1 + (2^q |y|)^{2n}} e^{-c(t-\tau)2^{2q}} \frac{1}{\sqrt{t}} \sum_{i=0}^{m+1} \left(|t^{\frac{i}{2}} \nabla^i u(x-y, \tau)|^2 + |t^{\frac{i}{2}} \nabla^i \nabla d(x-y, \tau)|^2 \right) dy d\tau \right\|_{L^\infty} dt \\
&\leq C_m 2^q \int_{2^{-2q}}^{\infty} \left\| \int_{\frac{t}{2}}^t \int_{|y| \leq 2t^{\frac{1}{2}}} \frac{2^{qn}}{1 + (2^q |y|)^{2n}} e^{-c(t-\tau)2^{2q}} \frac{1}{\sqrt{t}} \sum_{i=0}^{m+1} \left(|\tau^{\frac{i}{2}} \nabla^i u(x-y, \tau)|^2 + |\tau^{\frac{i}{2}} \nabla^i \nabla d(x-y, \tau)|^2 \right) dy d\tau \right\|_{L^\infty} dt \\
&\leq C_m 2^q \int_{2^{-2q}}^{\infty} \left\| \int_{\frac{t}{2}}^t \int_{|y| \leq 2t^{\frac{1}{2}}} \frac{2^{qn}}{1 + (2^q |y|)^{2n}} e^{-c(t-\tau)2^{2q}} \tau^{-\frac{3}{2}} \sum_{i=0}^{m+1} \left(\|t^{\frac{i}{2}} \nabla^i u\|_Z^2 + \|t^{\frac{i}{2}} \nabla^i d\|_X^2 \right) dy d\tau \right\|_{L^\infty} dt \\
&\leq C_m \varepsilon^2 2^q \int_{2^{-2q}}^{\infty} \int_{\frac{t}{2}}^t e^{-c(t-\tau)2^{2q}} \tau^{-\frac{3}{2}} d\tau dt \leq C_m \varepsilon^2.
\end{aligned}$$

Therefore, we get

$$2^q \int_{2^{-2q}}^{\infty} \|t^{\frac{m}{2}} \nabla^m \Delta_q \mathbb{T}_1(u, d)(\cdot, t)\|_{L^\infty} dt \leq C_m \varepsilon^2,$$

which along with (3.13) ensures that

$$\|t^{\frac{m}{2}} \nabla^m \Delta_q \mathbb{T}_1(u, d)\|_{\tilde{L}^1(\mathbb{R}_+; \dot{B}_{\infty, \infty}^1)} \leq C_m \varepsilon(1 + \varepsilon) \leq C_m \varepsilon,$$

if we choose ε small enough. The above inequality together with (3.12) gives the needed estimates of u .

To complete the proof of (3.10), it remains to prove that for any positive integers $m \geq 0$, there holds

$$\|t^{\frac{m}{2}} \nabla^{m+1} \mathbb{T}_2(u, d)(x, t)\|_{\tilde{L}^\infty(\mathbb{R}_+; \dot{B}_{\infty, \infty}^{-1}) \cap \tilde{L}^1(\mathbb{R}_+; \dot{B}_{\infty, \infty}^1)} \leq C_m \varepsilon. \quad (3.14)$$

For $t \in [2^{-2q}, \infty)$, by using (3.9) and Lemma 2.5, we have

$$\begin{aligned}
2^{-q} \|\Delta_q t^{\frac{m}{2}} \nabla^{m+1} \mathbb{T}_2(u, d)(\cdot, t)\|_{L^\infty} &\leq 2^{-q} \|t^{\frac{m}{2}} \nabla^{m+1} \mathbb{T}_2(u, d)(\cdot, t)\|_{L^\infty} \\
&= 2^{-q} \|t^{\frac{m}{2}} \nabla^m \nabla (d(\cdot, t) - (e^{t\Delta} d_0)(x))\|_{L^\infty} \\
&\leq 2^{-q} \|t^{\frac{m}{2}} \nabla^m \nabla d(\cdot, t)\|_{L^\infty} + 2^{-q} \|t^{\frac{m}{2}} \nabla^{m+1} e^{t\Delta} d_0\|_{L^\infty} \\
&\leq 2^{-q} t^{-\frac{1}{2}} \|t^{\frac{m}{2}} \nabla^m d\|_X + C_m [d_0]_{BMO} \\
&\leq C_m [d_0]_{BMO} \leq C_m \varepsilon,
\end{aligned}$$

where we have used (1.5) in the last inequality above. For $t \in (0, 2^{-2q})$, by denoting

$$\mathbf{G}(x, t) := (-u \cdot \nabla d + |\nabla d|^2 d)(x, t),$$

we can rewrite $\Delta_q \mathbb{T}_2(u, d)$ for any $q \in \mathbb{Z}$ as

$$\begin{aligned}
\Delta_q \mathbb{T}_2(u, d)(x, t) &= \int_0^{\frac{t}{2}} \int_{|y| \geq 2^{1-q}} g_{2,q}(y, t-\tau) \mathbf{G}(x-y, \tau) dy d\tau \\
&\quad + \int_{\frac{t}{2}}^t \int_{|y| \geq 2^{1-q}} g_{2,q}(y, t-\tau) \mathbf{G}(x-y, \tau) dy d\tau \\
&\quad + \int_0^{\frac{t}{2}} \int_{|y| \leq 2^{1-q}} g_{2,q}(y, t-\tau) \mathbf{G}(x-y, \tau) dy d\tau \\
&\quad + \int_{\frac{t}{2}}^t \int_{|y| \leq 2^{1-q}} g_{2,q}(y, t-\tau) \mathbf{G}(x-y, \tau) dy d\tau \\
&\triangleq (G_{1,q} + G_{2,q} + G_{3,q} + G_{4,q})(x, t),
\end{aligned}$$

where $g_{2,q}$ is given by Lemma 2.5. We shall estimate term by term from $G_{1,q}$ to $G_{3,q}$. Indeed thanks to (1.5) and Lemma 2.5, we have

$$|t^{\frac{m}{2}} \nabla^{m+1} G_{1,q}(x, t)| = \int_0^{\frac{t}{2}} \int_{|y| \geq 2^{1-q}} \left(\frac{t}{t-\tau} \right)^{\frac{m}{2}} (t-\tau)^{\frac{m}{2}} (\nabla^{m+1} g_{2,q})(y, t-\tau) \mathbf{G}(x-y, \tau) dy d\tau$$

$$\begin{aligned}
&\leq C_m \int_0^{\frac{t}{2}} \int_{|y| \geq 2^{1-q}} \frac{2^{qn}}{1 + (2^q|y|)^{2n}} e^{-c(t-\tau)2^{2q}} (t-\tau)^{-\frac{1}{2}} |\mathbf{G}(x-y, \tau)| dy d\tau \\
&\leq C_m \int_0^{\frac{t}{2}} \int_{|y| \geq 2^{1-q}} \frac{2^{qn}}{1 + (2^q|y|)^{2n}} e^{c\tau 2^{2q}} \tau^{-\frac{1}{2}} |\mathbf{G}(x-y, \tau)| dy d\tau \\
&\leq C_m \sum_{\ell \in \mathbb{Z}^n \setminus \{0\}} \int_0^t \int_{y \in 2^{-q}(\ell + [0,1]^n)} \frac{2^{q(n+1)}}{1 + (2^q|y|)^{2n}} |\mathbf{G}(x-y, \tau)| dy d\tau \\
&\leq C_m \sum_{\ell \in \mathbb{Z}^n \setminus \{0\}} \frac{2^{qn}}{|\ell|^{2n}} 2^q \int_0^{2^{-2q}} \int_{y \in 2^{-q}(\ell + [0,1]^n)} (|u(x-y, \tau)|^2 + |\nabla d(x-y, \tau)|^2) dy d\tau \\
&\leq C_m 2^q (\|u\|_Z^2 + \|d\|_X^2) \leq C_m 2^q \varepsilon^2,
\end{aligned}$$

where we have used the fact that $|d| = 1$ and $e^{c\tau 2^{2q}} \tau^{-\frac{1}{2}} \leq C 2^q$ for all $0 < \tau < \infty$. Exactly following the same line, we have

$$\begin{aligned}
&|t^{\frac{m}{2}} \nabla^{m+1} G_{2,q}(x, t)| \\
&= \int_{\frac{t}{2}}^t \int_{|y| \geq 2^{1-q}} g_{2,q}(y, t-\tau) t^{\frac{m}{2}} \nabla^{m+1} \mathbf{G}(x-y, \tau) dy d\tau \\
&\leq C_m \int_{\frac{t}{2}}^t \int_{|y| \geq 2^{1-q}} \frac{2^{qn}}{1 + (2^q y)^{2n}} e^{-c(t-\tau)2^{2q}} \frac{1}{\sqrt{t}} \sum_{i=0}^{m+1} \left(|t^{\frac{i}{2}} \nabla^i u(x-y, \tau)|^2 + |t^{\frac{i}{2}} \nabla^i \nabla d(x-y, \tau)|^2 |t^{\frac{i}{2}} \nabla^i d(x-y, \tau)| \right) dy d\tau \\
&\leq C_m \int_{\frac{t}{2}}^t \int_{|y| \geq 2^{1-q}} \frac{2^{q(n+1)}}{1 + (2^q y)^{2n}} \sum_{i=0}^{m+1} \left(|\tau^{\frac{i}{2}} \nabla^i u(x-y, \tau)|^2 + |\tau^{\frac{i}{2}} \nabla^i \nabla d(x-y, \tau)|^2 |\tau^{\frac{i}{2}} \nabla^i d(x-y, \tau)| \right) dy d\tau \\
&\leq C_m \sum_{\ell \in \mathbb{Z}^n \setminus \{0\}} \frac{2^{q(n+1)}}{|\ell|^{2n}} \left(1 + \sum_{i=1}^{m+1} \|t^{\frac{i}{2}} \nabla^i d(\cdot, t)\|_{L^\infty} \right) \int_0^{2^{-2q}} \int_{y \in 2^{-q}(\ell + [0,1]^n)} \sum_{i=0}^{m+1} \left(|\tau^{\frac{i}{2}} \nabla^i u(x-y, \tau)|^2 + |\tau^{\frac{i}{2}} \nabla^i \nabla d(x-y, \tau)|^2 \right) dy d\tau \\
&\leq C_m 2^{q(n+1)} \left(1 + \sum_{i=0}^m \|t^{\frac{i}{2}} \nabla^i d\|_X \right) \cdot 2^{-qn} \sum_{i=0}^{m+1} \left(\|\tau^{\frac{i}{2}} \nabla^i u\|_Z^2 + \|\tau^{\frac{i}{2}} \nabla^i d\|_X^2 \right) \\
&\leq C_m 2^q \varepsilon^2 (1 + \varepsilon); \\
&|t^{\frac{m}{2}} \nabla^{m+1} G_{3,q}(x, t)| \\
&= \int_0^{\frac{t}{2}} \int_{|y| \leq 2^{1-q}} \left(\frac{t}{t-\tau} \right)^{\frac{m}{2}} (t-\tau)^{\frac{m}{2}} (\nabla^{m+1} g_{2,q})(y, t-\tau) \mathbf{G}(x-y, \tau) dy d\tau \\
&\leq C_m \int_0^{\frac{t}{2}} \int_{|y| \leq 2^{1-q}} \frac{2^{qn}}{1 + (2^q|y|)^{2n}} e^{-c(t-\tau)2^{2q}} \frac{1}{\sqrt{t-\tau}} |\mathbf{G}(x-y, \tau)| dy d\tau \\
&\leq C_m \int_0^{\frac{t}{2}} \int_{|y| \leq 2^{1-q}} \frac{2^{q(n+1)}}{1 + (2^q|y|)^{2n}} |\mathbf{G}(x-y, \tau)| dy d\tau \\
&\leq C_m 2^{q(n+1)} \int_0^t \int_{|z-x| \leq 2^{1-q}} |\mathbf{G}(z, \tau)| dz d\tau \\
&\leq C_m 2^{q(n+1)} \int_0^{2^{-2q}} \int_{|z-x| \leq 2^{1-q}} (|u(z, \tau)|^2 + |\nabla d(z, \tau)|^2) dz d\tau \\
&\leq C_m 2^{q(n+1)} 2^{-qn} (\|u\|_Z^2 + \|d\|_X^2) \leq C_m 2^q \varepsilon^2,
\end{aligned}$$

and

$$\begin{aligned}
&|t^{\frac{m}{2}} \nabla^{m+1} G_{4,q}(x, t)| = \int_{\frac{t}{2}}^t \int_{|y| \leq 2^{1-q}} g_{2,q}(y, t-\tau) t^{\frac{m}{2}} \nabla^{m+1} \mathbf{G}(x-y, \tau) dy d\tau \\
&\leq C_m \int_{\frac{t}{2}}^t \int_{|y| \leq 2^{1-q}} \frac{2^{qn}}{1 + (2^q y)^{2n}} e^{-c(t-\tau)2^{2q}} \frac{1}{\sqrt{t}} \sum_{i=0}^{m+1} \left(|t^{\frac{i}{2}} \nabla^i u(x-y, \tau)|^2 + |t^{\frac{i}{2}} \nabla^i \nabla d(x-y, \tau)|^2 |t^{\frac{i}{2}} \nabla^i d(x-y, \tau)| \right) dy d\tau \\
&\leq C_m \int_{\frac{t}{2}}^t \int_{|y| \leq 2^{1-q}} \frac{2^{q(n+1)}}{1 + (2^q y)^{2n}} \sum_{i=0}^{m+1} \left(|\tau^{\frac{i}{2}} \nabla^i u(x-y, \tau)|^2 + |\tau^{\frac{i}{2}} \nabla^i \nabla d(x-y, \tau)|^2 |\tau^{\frac{i}{2}} \nabla^i d(x-y, \tau)| \right) dy d\tau
\end{aligned}$$

$$\begin{aligned}
&\leq C_m 2^{q(n+1)} \sum_{i=0}^{m+1} \int_0^{2^{-2q}} \int_{|x-z| \leq 2^{1-q}} \left(|\tau^{\frac{i}{2}} \nabla^i u(z, \tau)|^2 + |\tau^{\frac{i}{2}} \nabla^i \nabla d(z, \tau)|^2 \right) dz d\tau \cdot \left(1 + \sum_{i=1}^{m+1} \|t^{\frac{i}{2}} \nabla^i d(\cdot, t)\|_{L^\infty} \right) \\
&\leq C_m 2^q \left(1 + \sum_{i=0}^m \|t^{\frac{i}{2}} \nabla^i d\|_Z \right) \sum_{i=0}^{m+1} \left(\|\tau^{\frac{i}{2}} \nabla^i u\|_Z^2 + \|\tau^{\frac{i}{2}} \nabla^i d\|_X^2 \right) \leq C_m 2^q \varepsilon^2 (1 + \varepsilon),
\end{aligned}$$

where we have used the fact that $|d| = 1$ in the above inequalities. As a consequence, we obtain

$$\|t^{\frac{m}{2}} \nabla^{m+1} \mathbb{T}_2(u, d)(x, t)\|_{\tilde{L}^\infty(\mathbb{R}_+; \dot{B}_{\infty, \infty}^{-1})} \leq C_m \varepsilon (1 + \varepsilon (1 + \varepsilon)) \leq C_m \varepsilon, \quad (3.15)$$

if we selecting ε small enough. On the other hand, we have

$$\begin{aligned}
&2^q \int_0^{2^{-2q}} \|t^{\frac{m}{2}} \nabla^{m+1} \Delta_q \mathbb{T}_2(u, d)(\cdot, t)\|_{L^\infty} dt \\
&\leq 2^q \int_0^{2^{-2q}} \|t^{\frac{m}{2}} \nabla^m \nabla d(\cdot, t)\|_{L^\infty} dt + 2^q \int_0^{2^{-2q}} \|t^{\frac{m}{2}} \nabla^{m+1} e^{t\Delta} d_0\|_{L^\infty} dt \\
&\leq 2^q \int_0^{2^{-2q}} t^{-\frac{1}{2}} dt \|t^{\frac{m}{2}} \nabla^m d\|_X + C_m [d_0]_{BMO} \leq C_m \varepsilon,
\end{aligned} \quad (3.16)$$

Now let us turn to the estimate of $\int_{2^{-2q}}^\infty \|t^{\frac{m}{2}} \nabla^{m+1} \Delta_q \mathbb{T}_2(u, d)(\cdot, t)\|_{L^\infty} dt$. We split $\Delta_q \mathbb{T}_2(u, d)$ for all $q \in \mathbb{Z}$ as

$$\begin{aligned}
\Delta_q \mathbb{T}_1(u, d)(x, t) &= \int_0^{\frac{t}{2}} \Delta_q e^{(t-\tau)\Delta} \mathbf{G}(x, \tau) d\tau + \int_{\frac{t}{2}}^t \int_{|y| \geq 2\sqrt{t}} g_{2,q}(y, t-\tau) \mathbf{G}(x-y, \tau) dy d\tau \\
&\quad + \int_{\frac{t}{2}}^t \int_{|y| \leq 2\sqrt{t}} g_{2,q}(y, t-\tau) \mathbf{G}(x-y, \tau) dy d\tau \\
&:\triangleq (\tilde{G}_{1,q} + \tilde{G}_{2,q} + \tilde{G}_{3,q})(x, t).
\end{aligned}$$

We shall estimate term by term from $\tilde{G}_{1,q}$ to $\tilde{G}_{3,q}$. Indeed, thanks to (3.9) and Lemma 2.5, we have

$$\begin{aligned}
&2^q \int_{2^{-2q}}^\infty \|t^{\frac{m}{2}} \nabla^{m+1} \tilde{G}_{1,q}(\cdot, t)\|_{L^\infty} dt \\
&= 2^q \int_{2^{-2q}}^\infty \left\| t^{\frac{m}{2}} \nabla^{m+1} \Delta_q e^{\frac{t}{2}\Delta} \left(d(x, \frac{t}{2}) - (e^{\frac{t}{2}\Delta} d_0)(x) \right) \right\|_{L^\infty} dt \\
&\leq 2^q \int_{2^{-2q}}^\infty \left(\left\| t^{\frac{m}{2}} \nabla^{m+1} \Delta_q e^{\frac{t}{2}\Delta} d(x, \frac{t}{2}) \right\|_{L^\infty} + \left\| t^{\frac{m}{2}} \nabla^{m+1} \Delta_q (e^{t\Delta} d_0)(x) \right\|_{L^\infty} \right) dt \\
&\leq C_m 2^q \int_{2^{-2q}}^\infty \left(\left\| \int_{\mathbb{R}^n} g_{3,q}(y, \frac{t}{2}) \cdot \nabla d(x-y, \frac{t}{2}) dy \right\|_{L^\infty} + \|g_{3,q} * \Delta_q \nabla d_0\|_{L^\infty} \right) dt \\
&\leq C_m 2^q \int_{2^{-2q}}^\infty \int_{\mathbb{R}^n} \frac{2^{qn}}{1 + (2^q y)^{2n}} e^{-ct2^{2q}} dy \left(\left\| \nabla d(\cdot, \frac{t}{2}) \right\|_{L^\infty} + 2^q \|\Delta_q d_0\|_{L^\infty} \right) dt \\
&\leq C_m 2^q \int_{2^{-2q}}^\infty e^{-ct2^{2q}} t^{-\frac{1}{2}} dt \|d\|_X + C_m 2^{2q} \int_{2^{-2q}}^\infty e^{-ct2^{2q}} dt \|\Delta_q d_0\|_{L^\infty} \\
&\leq C_m \|d\|_X + C_m \|\Delta_q d_0\|_{L^\infty} \leq C_m (\|d\|_X + \|d_0\|_{\dot{B}_{\infty, \infty}^0}) \\
&\leq C_m \varepsilon + C_m [u_0]_{BMO} \leq C_m \varepsilon,
\end{aligned}$$

where we have used the estimate (1.5) and the embedding $BMO \hookrightarrow \dot{B}_{\infty, \infty}^0$. Exactly following the same line, we have

$$\begin{aligned}
&2^q \int_{2^{-2q}}^\infty \|t^{\frac{m}{2}} \nabla^{m+1} \tilde{G}_{2,q}(\cdot, t)\|_{L^\infty} dt \\
&\leq C_m 2^q \int_{2^{-2q}}^\infty \left\| \int_{\frac{t}{2}}^t \int_{|y| \geq 2t^{\frac{1}{2}}} \frac{2^{qn}}{1 + (2^q |y|)^{2n}} e^{-c(t-\tau)2^{2q}} |t^{\frac{m}{2}} \nabla^{m+1} \mathbf{G}(x-y, \tau)| dy d\tau \right\|_{L^\infty} dt \\
&\leq C_m 2^q \int_{2^{-2q}}^\infty \left\| \int_{\frac{t}{2}}^t \int_{|y| \geq 2t^{\frac{1}{2}}} \frac{2^{qn}}{1 + (2^q |y|)^{2n}} \frac{1}{\sqrt{t}} \sum_{i=0}^{m+1} \left(|t^{\frac{i}{2}} \nabla^i u(x-y, \tau)|^2 + |t^{\frac{i}{2}} \nabla^i \nabla d(x-y, \tau)|^2 + |t^{\frac{i}{2}} \nabla^i d(x-y, \tau)|^2 \right) dy d\tau \right\|_{L^\infty} dt
\end{aligned}$$

$$\begin{aligned}
&\leq C_m 2^q \sum_{\ell \in \mathbb{Z}^n \setminus \{0\}} \int_{2^{-2q}}^{\infty} \left\| \int_{\frac{t}{2}}^t \int_{|y| \in \sqrt{t}(\ell + [0,1]^n)} \frac{2^{qn}}{1 + (2^q |y|)^{2n}} \frac{1}{\sqrt{t}} \right. \\
&\quad \times \sum_{i=0}^{m+1} \left(|\tau^{\frac{i}{2}} \nabla^i u(x-y, \tau)|^2 + |\tau^{\frac{i}{2}} \nabla^i \nabla d(x-y, \tau)|^2 |\tau^{\frac{i}{2}} \nabla^i d(x-y, \tau)| \right) dy d\tau \Big\|_{L^\infty} dt \\
&\leq C_m 2^q \sum_{\ell \in \mathbb{Z}^n \setminus \{0\}} \int_{2^{-2q}}^{\infty} \left\| \frac{2^{-qn}}{(|\ell| \sqrt{t})^{2n}} \frac{1}{\sqrt{t}} \left(1 + \sum_{i=1}^{m+1} \|t^{\frac{i}{2}} \nabla^i d(\cdot, t)\|_{L^\infty} \right) \right. \\
&\quad \times \int_{\frac{t}{2}}^t \int_{|y| \in \sqrt{t}(\ell + [0,1]^n)} \sum_{i=0}^{m+1} \left(|\tau^{\frac{i}{2}} \nabla^i u(x-y, \tau)|^2 + |\tau^{\frac{i}{2}} \nabla^i \nabla d(x-y, \tau)|^2 \right) dy d\tau \Big\|_{L^\infty} dt \\
&\leq C_m 2^q \sum_{\ell \in \mathbb{Z}^n \setminus \{0\}} \int_{2^{-2q}}^{\infty} \left(\frac{2^{-qn}}{(|\ell| \sqrt{t})^{2n}} t^{\frac{n-1}{2}} \left(1 + \sum_{i=0}^m \|t^{\frac{i}{2}} \nabla^i d\|_X \right) \sum_{i=0}^{m+1} \left(\|\tau^{\frac{i}{2}} \nabla^i u\|_Z^2 + \|\tau^{\frac{i}{2}} \nabla^i d\|_X^2 \right) \right) dt \\
&\leq C_m \sum_{\ell \in \mathbb{Z}^n \setminus \{0\}} \frac{1}{|\ell|^{2n}} \left(1 + \sum_{i=0}^m \|t^{\frac{i}{2}} \nabla^i d\|_X \right) \sum_{i=0}^{m+1} \left(\|\tau^{\frac{i}{2}} \nabla^i u\|_Z^2 + \|\tau^{\frac{i}{2}} \nabla^i d\|_X^2 \right) \leq C_m \varepsilon^2 (1 + \varepsilon),
\end{aligned}$$

and

$$\begin{aligned}
&2^q \int_{2^{-2q}}^{\infty} \|t^{\frac{m}{2}} \nabla^{m+1} \tilde{G}_{3,q}(\cdot, t)\|_{L^\infty} dt \\
&\leq C_m 2^q \int_{2^{-2q}}^{\infty} \left\| \int_{\frac{t}{2}}^t \int_{|y| \leq 2t^{\frac{1}{2}}} \frac{2^{qn}}{1 + (2^q |y|)^{2n}} e^{-c(t-\tau)2^{2q}} |t^{\frac{m}{2}} \nabla^{m+1} \mathbf{G}(x-y, \tau)| dy d\tau \right\|_{L^\infty} dt \\
&\leq C_m 2^q \int_{2^{-2q}}^{\infty} \left\| \int_{\frac{t}{2}}^t \int_{|y| \leq 2t^{\frac{1}{2}}} \frac{2^{qn}}{1 + (2^q |y|)^{2n}} \frac{1}{\sqrt{t}} \sum_{i=0}^{m+1} \left(|\tau^{\frac{i}{2}} \nabla^i u(x-y, \tau)|^2 + |\tau^{\frac{i}{2}} \nabla^i \nabla d(x-y, \tau)|^2 |\tau^{\frac{i}{2}} \nabla^i d(x-y, \tau)| \right) dy d\tau \right\|_{L^\infty} dt \\
&\leq C_m 2^q \int_{2^{-2q}}^{\infty} \left\| \int_{\frac{t}{2}}^t \int_{|y| \leq 2t^{\frac{1}{2}}} \frac{2^{qn}}{1 + (2^q |y|)^{2n}} \frac{1}{\sqrt{t}} \sum_{i=0}^{m+1} \left(|\tau^{\frac{i}{2}} \nabla^i u(x-y, \tau)|^2 + |\tau^{\frac{i}{2}} \nabla^i \nabla d(x-y, \tau)|^2 |\tau^{\frac{i}{2}} \nabla^i d(x-y, \tau)| \right) dy d\tau \right\|_{L^\infty} dt \\
&\leq C_m 2^q \int_{2^{-2q}}^{\infty} \left\| \int_{\frac{t}{2}}^t \int_{|y| \leq 2\sqrt{t}} \frac{2^{qn}}{1 + (2^q |y|)^{2n}} \frac{1}{\sqrt{\tau}} \sum_{i=0}^{m+1} \left(1 + \|\tau^{\frac{i}{2}} \nabla^i d(\cdot, \tau)\|_{L^\infty} \right) \left(|\tau^{\frac{i}{2}} \nabla^i u(x-y, \tau)|^2 + |\tau^{\frac{i}{2}} \nabla^i \nabla d(x-y, \tau)|^2 \right) dy d\tau \right\|_{L^\infty} dt \\
&\leq C_m 2^q \int_{2^{-2q}}^{\infty} \left\| \int_{\frac{t}{2}}^t \int_{|y| \leq 2\sqrt{t}} \frac{2^{qn}}{1 + (2^q |y|)^{2n}} \tau^{-\frac{3}{2}} \sum_{i=0}^{m+1} \left(1 + \|\tau^{\frac{i}{2}} \nabla^i d\|_X \right) \left(\|\tau^{\frac{i}{2}} \nabla^i u\|_Z^2 + \|\tau^{\frac{i}{2}} \nabla^i d\|_X^2 \right) dy d\tau \right\|_{L^\infty} dt \\
&\leq C_m 2^q \int_{2^{-2q}}^{\infty} \int_{\frac{t}{2}}^t e^{-c(t-\tau)2^{2q}} \tau^{-\frac{3}{2}} d\tau dt \sum_{i=0}^{m+1} \left(1 + \|\tau^{\frac{i}{2}} \nabla^i d\|_X \right) \left(\|\tau^{\frac{i}{2}} \nabla^i u\|_Z^2 + \|\tau^{\frac{i}{2}} \nabla^i d\|_X^2 \right) \\
&\leq C_m \sum_{i=0}^{m+1} \left(1 + \|\tau^{\frac{i}{2}} \nabla^i d\|_X \right) \left(\|\tau^{\frac{i}{2}} \nabla^i u\|_Z^2 + \|\tau^{\frac{i}{2}} \nabla^i d\|_X^2 \right) \leq C_m \varepsilon^2 (1 + \varepsilon).
\end{aligned}$$

Therefore, by selecting ε small enough, we obtain

$$2^q \int_{2^{-2q}}^{\infty} \|t^{\frac{m}{2}} \nabla^{m+1} \Delta_q \mathbb{T}_2(u, d)(\cdot, t)\|_{L^\infty} dt \leq C_m \varepsilon (1 + \varepsilon(1 + \varepsilon)) \leq C_m \varepsilon,$$

which along with (3.16) ensure that

$$\|t^{\frac{m}{2}} \nabla^{m+1} \Delta_q \mathbb{T}_2(u, d)\|_{\tilde{L}^1(\mathbb{R}_+; \dot{B}_{\infty, \infty}^1)} \leq C_m \varepsilon.$$

Combining the inequality above and (3.15) together, we complete the proof of (3.14).

Step 2. Estimate (1.6) for all $k, m \geq 0$.

Subsequently, we shall prove (1.6) for all $k, m \geq 0$ by the standard induction method. Let $k > 0$ be a fixed positive integer, by using the results of **Step 1**, we may assume that (1.6) holds for all $0 \leq \ell \leq k-1$ and for all $m \geq 0$, i.e., there hold

$$\|t^{\frac{m}{2} + \ell} (\partial_t^\ell \nabla^m u, \partial_t^\ell \nabla^{m+1} d)\|_{\tilde{L}^\infty(\mathbb{R}_+; \dot{B}_{\infty, \infty}^{-1}) \cap \tilde{L}^1(\mathbb{R}_+; \dot{B}_{\infty, \infty}^1)} \leq C_{m, k} \varepsilon \quad \forall m \geq 0 \text{ and } \ell = 1, 2, \dots, k-1. \quad (3.17)$$

In what follows, we shall have completed the proof of Theorem 1.3 if we prove that (3.17) still holds for k . Before going to do it, we notice that by using (3.17) and the fact that the operator ∇^{-1} is bounded from $\dot{B}_{\infty,\infty}^{-s-1}$ to $\dot{B}_{\infty,\infty}^{-s}$ for all $s \geq 0$, we have

$$\begin{aligned} \|t^{\frac{m}{2}+\ell} (\partial_t^\ell \nabla^{m+1} u, \partial_t^\ell \nabla^{m+1} \nabla d) \|_{\tilde{L}^\infty(\mathbb{R}_+; \dot{B}_{\infty,\infty}^0)} &\leq \|t^{\frac{m}{2}+\ell} (\partial_t^\ell \nabla^m u, \partial_t^\ell \nabla^m \nabla d) \|_{\tilde{L}^\infty(\mathbb{R}_+; \dot{B}_{\infty,\infty}^{-1})} \\ &\leq C_{m,k} \varepsilon \quad \forall m \geq 0 \text{ and } \ell = 1, 2, \dots, k-1. \end{aligned} \quad (3.18)$$

By using the standard interpolation theory, it follows from (3.17) that

$$\| (t^{\frac{m}{2}+\ell} \partial_t^\ell \nabla^m u, \partial_t^\ell \nabla^m \nabla d) \|_{\tilde{L}^2(\mathbb{R}_+; \dot{B}_{\infty,\infty}^0)} \leq C_{m,k} \varepsilon \quad \forall m \geq 0 \text{ and } \ell = 1, 2, \dots, k-1. \quad (3.19)$$

We also notice that when we study the linear nonhomogeneous fractional heat equation

$$\partial_t \Psi - \Delta \Psi = F.$$

For any $k \geq 1$, by induction we have

$$\partial_t^k \Psi = \Delta^k \Psi + \sum_{i=0}^{k-1} \Delta^{k-1-i} \partial_t^i F. \quad (3.20)$$

On the other hand, we can rewrite (1.1) as

$$\partial_t u - \Delta u = -\mathbb{P} \nabla \cdot (u \otimes u + \nabla d \odot \nabla d).$$

Hence, by using the inequality (3.20), and by writing $Y := \tilde{L}^\infty(\mathbb{R}_+; \dot{B}_{\infty,\infty}^{-1}) \cap \tilde{L}^1(\mathbb{R}_+; \dot{B}_{\infty,\infty}^1)$, we have

$$\begin{aligned} \|t^{\frac{m}{2}+k} \partial_t^k \nabla^m u\|_Y &\leq \|t^{\frac{m}{2}+k} \Delta^k \nabla^m u\|_Y + \sum_{i=0}^{k-1} \|t^{\frac{m}{2}+k} \Delta^{k-1-i} \partial_t^i \nabla^m (\mathbb{P} \nabla \cdot (u \otimes u + \nabla d \odot \nabla d))\|_Y \\ &\leq \|t^{\frac{m}{2}+k} \nabla^{2k+m} u\|_Y + \sum_{i=0}^{k-1} \|t^{\frac{m}{2}+k} \partial_t^i \nabla^{2(k-1-i)+m+1} (u \otimes u + \nabla d \odot \nabla d)\|_Y \\ &:= II_1 + II_2. \end{aligned} \quad (3.21)$$

Repeating the progress as in derive (3.17) and (3.20), it is easy to see

$$II_1 \leq C_{m,k} \varepsilon (1 + \varepsilon)$$

For the rest term II_2 , by using Lemma 2.4, the inequalities (3.18) and (3.19), we have

$$\begin{aligned} II_2 &\leq \sum_{i=0}^{k-1} \|t^{\frac{m}{2}+k} \partial_t^i \nabla^{2(k-1-i)+m+1} (u \otimes u + \nabla d \odot \nabla d)\|_{\tilde{L}^2(\mathbb{R}_+; \dot{B}_{\infty,\infty}^0)} \\ &\leq \sum_{i=0}^{k-1} \sum_{\ell=0}^{2(k-1-i)+m+1} \sum_{\ell_1=0}^{\ell} \left(\left\| t^{\frac{\ell}{2}+\ell_1} \partial_t^{\ell_1} \nabla^\ell u \cdot |t^{\frac{m-\ell}{2}+k-\ell_1} \partial_t^{i-\ell_1} \nabla^{2(k-1-i)+m+1-\ell} u| \right\|_{\tilde{L}^2(\mathbb{R}_+; \dot{B}_{\infty,\infty}^0)} \right. \\ &\quad \left. + \left\| t^{\frac{\ell}{2}+\ell_1} \partial_t^{\ell_1} \nabla^\ell \nabla d \cdot |t^{\frac{m-\ell}{2}+k-\ell_1} \partial_t^{i-\ell_1} \nabla^{2(k-1-i)+m+1-\ell} \nabla d| \right\|_{\tilde{L}^2(\mathbb{R}_+; \dot{B}_{\infty,\infty}^0)} \right) \\ &\leq C \sum_{i=0}^{k-1} \sum_{\ell=0}^{2(k-1-i)+m} \sum_{\ell_1=0}^{\ell} \left\| t^{\frac{\ell}{2}+\ell_1} \left(\partial_t^{\ell_1} \nabla^\ell u, \partial_t^{\ell_1} \nabla^\ell \nabla d \right) \right\|_{\tilde{L}^2(\mathbb{R}_+; \dot{B}_{\infty,\infty}^0)} \\ &\quad \times \left\| t^{\frac{m-\ell}{2}+k-\ell_1} \left(\partial_t^{i-\ell_1} \nabla^{2(k-1-i)+m-\ell+1} u, \partial_t^{i-\ell_1} \nabla^{2(k-1-i)+m-\ell+1} \nabla d \right) \right\|_{\tilde{L}^\infty(\mathbb{R}_+; \dot{B}_{\infty,\infty}^0)} \\ &\leq C_{m,k} \varepsilon^2. \end{aligned}$$

Inserting the estimates of II_1 and II_2 into (3.21), it follows that

$$\|t^{\frac{m}{2}+k} \partial_t^k \nabla^m u\|_{\tilde{L}^\infty(\mathbb{R}_+; \dot{B}_{\infty,\infty}^{-1}) \cap \tilde{L}^1(\mathbb{R}_+; \dot{B}_{\infty,\infty}^1)} \leq C_m \varepsilon,$$

if we choose ε small enough.

By rewriting (1.2) as

$$\partial_t(\nabla d) - \Delta \nabla d = -\nabla(u \cdot \nabla d + |\nabla d|^2 d),$$

and by repeating a similar process of the derivations of the velocity field u , we can handle the case for d , i.e., there holds

$$\|t^{\frac{m}{2}+k} \partial_t^k \nabla^m \nabla d\|_{\tilde{L}^\infty(\mathbb{R}_+; \dot{B}_{\infty,\infty}^{-1}) \cap \tilde{L}^1(\mathbb{R}_+; \dot{B}_{\infty,\infty}^1)} \leq C_m \varepsilon,$$

if we choose ε small enough. Therefore, we conclude that (1.6) is still holds for k . This completes the proof of Theorem 1.3. \square

Remark 3.2 *It is standard that the condition (1.4) is preserved by the flow. In fact, by applying the maximum principle to the equations of $|d|^2$, we can easily see that $|d| = 1$ under the initial assumption the $|d_0| = 1$. We omitted this step due to it is standard.*

4 Proof of Theorem 1.5

In this section, following the methods used by Chemin in [1, 2] (see also [41]), we give the proof of Theorem 1.5. We first notice that the proof of the existence of $\gamma(x, t)$ is exactly as the counterpart in Theorem 3.4 of [1] (or Theorem 3.2.1 of [2]), and we omit the details here. In what follows, the main issue is to prove (1.7). Indeed, for any $x_1, x_2 \in \mathbb{R}^n$ and $q \in \mathbb{Z}$, let us decompose $(u, \nabla d)$ in a low and a high frequency part. This leads to, for all $t \in \mathbb{R}_+$, we have

$$\begin{aligned} & |\gamma(x_1, t) - \gamma(x_2, t)| \\ & \leq |x_1 - x_2| + \int_0^t |(S_q u(\gamma(x_1, s), s) - S_q u(\gamma(x_2, s), s), S_q \nabla d(\gamma(x_1, s), s) - S_q \nabla d(\gamma(x_2, s), s))| ds \\ & \quad + 2 \int_0^t \sum_{p \geq q} \|(\Delta_p u(\cdot, s), \Delta_p \nabla d(\cdot, s))\|_{L^\infty} ds \\ & \leq |x_1 - x_2| + \int_0^t \|(\nabla S_q u(\cdot, s), \nabla S_q \nabla d(\cdot, s))\|_{L^\infty} |\gamma(x_1, s) - \gamma(x_2, s)| ds \\ & \quad + 2 \sum_{p \geq q} 2^{-p} \cdot \sup_{p \geq q} \left\{ \int_0^t 2^p \|(\Delta_p u(\cdot, s), \Delta_p \nabla d(\cdot, s))\|_{L^\infty} ds \right\} \\ & \leq |x_1 - x_2| + \int_0^t \|(\nabla S_q u(\cdot, s), \nabla S_q \nabla d(\cdot, s))\|_{L^\infty} |\gamma(x_1, s) - \gamma(x_2, s)| ds \\ & \quad + 2^{2-q} \|(u, \nabla d)\|_{\tilde{L}^\infty(\mathbb{R}_+; \dot{B}_{\infty,\infty}^1)}. \end{aligned}$$

Let $\rho(t) \triangleq |\gamma(x_1, t) - \gamma(x_2, t)|$ and

$$D_q \triangleq |x_1 - x_2| + 2^{2-q} \|(u, \nabla d)\|_{\tilde{L}^\infty(\mathbb{R}_+; \dot{B}_{\infty,\infty}^1)} + \int_0^t \|(\nabla S_q u(\cdot, s), \nabla S_q \nabla d(\cdot, s))\|_{L^\infty} |\gamma(x_1, s) - \gamma(x_2, s)| ds.$$

Then we have

$$\rho(t) \leq D_q(t) \quad \text{for all } q \in \mathbb{Z},$$

and

$$D_q(t) \leq |x_1 - x_2| + 2^{2-q} \|(u, \nabla d)\|_{\tilde{L}^\infty(\mathbb{R}_+; \dot{B}_{\infty,\infty}^1)} + \int_0^t \|(\nabla S_q u(\cdot, s), \nabla S_q \nabla d(\cdot, s))\|_{L^\infty} D_q(s) ds.$$

The Gronwall inequality implies that, for any $t > 0$,

$$\begin{aligned} D_q(t) &\leq (|x_1 - x_2| + 2^{2-q} \|(u, \nabla d)\|_{\tilde{L}^\infty(\mathbb{R}_+; \dot{B}_{\infty, \infty}^1)}) \exp \left\{ \int_0^t \|(\nabla S_q u(\cdot, s), \nabla S_q \nabla d(\cdot, s))\|_{L^\infty} ds \right\} \\ &\leq (|x_1 - x_2| + C2^{2-q} \varepsilon) \exp \left\{ \int_0^t \|\nabla S_q u(\cdot, s)\|_{L^\infty} ds \right\}, \end{aligned} \quad (4.1)$$

where we have used (1.6) with $k = m = 0$ in the last inequality above. Notice that the above inequality holds for all $q \in \mathbb{Z}$, and by selecting $q \geq 1$, we have

$$\begin{aligned} &\int_0^t \|(\nabla S_q u(\cdot, s), \nabla S_q \nabla d(\cdot, s))\|_{L^\infty} ds \\ &\leq \int_0^t \sum_{p \leq 0} 2^p \|(\Delta_p u(\cdot, s), \Delta_p \nabla d(\cdot, s))\|_{L^\infty} ds + \sum_{p=0}^q \int_0^t 2^p \|(u(\cdot, s), \nabla d(\cdot, s))\|_{L^\infty} ds \\ &\leq \int_0^t \sum_{p \leq 0} 2^{2p} \cdot 2^{-p} \|(\Delta_p u(\cdot, s), \Delta_p \nabla d(\cdot, s))\|_{L^\infty} ds + q \|(u, \nabla d)\|_{\tilde{L}_t^1(\dot{B}_{\infty, \infty}^1)} \\ &\leq Ct \|(u, \nabla d)\|_{\tilde{L}^\infty(\mathbb{R}_+; \dot{B}_{\infty, \infty}^{-1})} + q \|(u, \nabla d)\|_{\tilde{L}^1(\mathbb{R}_+; \dot{B}_{\infty, \infty}^1)} \\ &\leq C(t + q) \|(u, \nabla d)\|_{\tilde{L}^\infty(\mathbb{R}_+; \dot{B}_{\infty, \infty}^{-1}) \cap \tilde{L}^1(\mathbb{R}_+; \dot{B}_{\infty, \infty}^1)} \leq C\varepsilon(t + q). \end{aligned}$$

The above inequality along with (4.1) implies that

$$D_q(t) \leq (|x_1 - x_2| + C2^{2-q} \varepsilon) \exp \{C\varepsilon(t + q)\}.$$

By choosing $2^q \equiv |x_1 - x_2|^{-1}$ in the above inequality, we infer that (1.7) holds, and the Theorem 1.5 is proved. \square

References

- [1] J. Y. Chemin, Le système of Navier–Stokes incompressible soixante dix ans après Jean Leray, Séminaires & Congrès, **9** (2004), 99–123.
- [2] J. Y. Chemin, Localization in Fourier space and Navier–Stokes system, 2004. <http://www.math.uzh.ch/pde13/fileadmin/pde13/pdf/coursPisa.pdf>.
- [3] M. Cannone, A generalization of a theorem by Kato on Navier–Stokes equations, Revista Mate. Ibero., **13** (1997), 515–541.
- [4] H. Dong and D. Du, On the local smoothness of solutions of the Navier–Stokes equations, J. Math. Fluid Mech., **9** (2007), 139–152.
- [5] H. Dong and D. Li, Optimal local smoothing and analyticity rate estimates for the generalized Navier–Stokes equations, Commun. Math. Sci., **7** (2009), 67–80.
- [6] Y. Du, Space-time regularity of the Koch&Tataru solutions to Navier–Stokes equations, Nonlinear Anal., **104** (2014), 124–132.
- [7] Y. Du and K. Wang, Space-time regularity of the Kock & Tataru solutions to the liquid crystal equations, SIAM J. Math. Anal., **45**(6), 3838–3853.
- [8] Y. Du and K. Wang, Regularity of the solutions to the liquid crystal equations with small rough data, J. Differ. Equ., **256** (2014), 65–81.

- [9] J. L. Ericksen, Hydrostatic theory of liquid crystal, Arch. Rational Mech. Anal., **9** (1962), 371–378.
- [10] H. Fujita and T. Kato, On the Navier–Stokes initial value problem, I, Arch. Rational Mech. Anal., **16** (1964), 269–315.
- [11] P. Germain, N. Pavlović and G. Staffilani, Regularity of solutions to the Navier-Stokes equations evolving from small data in BMO^{-1} , Int. Math. Res. Not., **21** (2007), 35pp. Art.ID rnm087.
- [12] Y. Giga and O. Sawada, On regularizing-decay rate estimates for solutions to the Navier-Stokes initial value problem, Nonlinear Anal. Appl., **1**(2) (2003), 549–562.
- [13] J. Hineman and C. Wang, Well-posedness of nematic liquid crystal flow in $L^3_{loc}(\mathbb{R}^3)$, Arch. Rational Mech. Anal., **210** (2013), 177–218.
- [14] M. Hong, Global existence of solutions of the simplified Ericksen–Leslie system in dimension two, Cal. Var., **40** (2011), 15–36.
- [15] X. Hu and D. Wang, Global solution to the three-dimensional incompressible flow of liquid crystals, Commun. Math. Phys., **296** (2010), 861–880.
- [16] T. Huang and C. Wang, Blow up criterion for nematic liquid crystal flows, Comm. Partial Differ. Equ., **37** (2012), 875–884.
- [17] Y. Giga, Solutions for semilinear parabolic equations in L^p and regularity of weak solutions of the Navier–Stokes system, J. Differ. Equ., **61** (1986), 186–212.
- [18] T. Kato, Strong L^p -solutions of the Navier-Stokes equations in \mathbf{R}^m , with applications to weak solutions, Math. Z., **187** (1984), 471–480.
- [19] T. Kato and H. Fujita, On the non-stationary Navier–Stokes system, Rendicanti del seminaria Matematico della Università di Padova, **30** (1962), 243–260.
- [20] H. Koch and D. Tataru, Well-posedness for the Navier–Stokes equations, Adv. Math., **157** (2001), 22–35.
- [21] F. Leslie, Theory of flow phenomenon in liquid crystals. In: The Theory of Liquid Crystals, London-New York: Academic Press, **4** (1979), 1–81.
- [22] P.G. Lemarié-Rieusset, *Recent Developments in the Navier-Stokes Problem*, Chapman and Hall/CRC, 2002.
- [23] J. Leray, Sur le mouvement d’un liquide visqueux emplissant l’espace, Acta. Math., **63** (1934), 193–248.
- [24] X. Li and D. Wang, Global solution to the incompressible flow of liquid crystal, J. Differ. Equ., **252** (2012), 745–767.
- [25] F. Lin, Nonlinear theory of defects in nematic liquid crystals; phase transition and flow phenomena, Commun. Pure Appl. Math., **42** (1989), 789–814.
- [26] F. Lin, J. Lin and C. Wang, Liquid crystal flow in two dimensions, Arch. Ration. Mech. Anal., **197** (2010), 297–336.
- [27] F. Lin and C. Liu, Nonparabolic dissipative systems modeling the flow of liquid crystals, Commun. Pure Appl. Math., **48** (1995), 501–537.
- [28] F. Lin and C. Liu, Partial regularities of the nonlinear dissipative systems modeling the flow of liquid crystals, Disc. Contin. Dyn. Syst., **A 2** (1996), 1–23.

- [29] F. Lin and C. Wang, On the uniqueness of heat flow of harmonic maps and hydrodynamic flow of nematic liquid crystals, *Chinese Annal. Math. Ser. B*, **31** (2010), 921–938.
- [30] J. Lin and S. Ding, On the well-posedness for the heat flow of harmonic maps and the hydrodynamic flow of nematic liquid crystals in critical spaces, *Math. Meth. Appl. Sci.*, **35** (2012), 158–173.
- [31] C. Liu and N. J. Wakington, Approximation of liquid crystal flows, *SIAM J. Numer. Anal.*, **37** (2000), 725–741.
- [32] Q. Liu, Well-posedness for the nematic liquid crystal flow with rough initial data, To appear *Chinese Annal. Math. Ser. A*.
- [33] H. Miura and O. Sawada, On the regularizing rate estimates of Koch-Tataru’s solution to the Navier-Stokes equations, *Asymptotic Anal.*, **49** (2006), 1–15.
- [34] O. Sawada, On analyticity rate estimates of the solutions to the Navier-Stokes equations in Bessel-potential spaces, *J. Math. Anal. Appl.*, **312** (2005), 1–13.
- [35] M. Struwe, On the evolution of harmonic maps of Riemannian surfaces. *Comment. Math. Helv.*, **60** (1985), 558–581.
- [36] I. W. Stewart, *The static and dynamic continuum theory of liquid crystals*, Taylor & Francis, London and New York, 2004.
- [37] H. Sun and C. Liu, On energetic variational approaches in modeling the nematic liquid crystal flows, *Disc. Contin. Dyn. Syst., A* **23** (2009), 455–475.
- [38] C. Wang, Well-posedness for the heat flow of harmonic maps and the liquid crystal flow with rough initial data, *Arch. Rational Mech. Anal.*, **200** (2011), 1–19.
- [39] H. Wen and S. Ding, Solutions of incompressible hydrodynamic flow of liquid crystals, *Nonlinear Anal. Real Word Appl.*, **12** (2011), 1510–1531.
- [40] X. Xu and Z. Zhang, Global regularity and uniqueness of weak solution for the 2-D liquid crystal flows, *J. Differ. Equ.*, **252** (2012), 1169–1181.
- [41] P. Zhang and T. Zhang, Regularity of the Koch-Tataru solutions to Navier–Stokes system, *Sci. China Math.*, **55**(2) (2012), 453–464.